14.23

Instead of a single charge e moving with constant velocity $\omega_0 R$ in a circular path of radius R , as in Problem 14.15, N charges q_j move with fixed relative positions θ_j around the same circle.

(a) Show that the power radiated into the mth multiple of ω_0 is

$$\frac{dP_m(N)}{d\Omega} = \frac{dP_m(1)}{d\Omega} F_m(N)$$

where $\frac{dP_m(1)}{d\Omega}$ is the result of part a in Problem 14.15 with $e \longrightarrow 1$, and

$$F_m(N) = |\sum_{j=1}^{N} q_j e^{\imath m\theta_j}|^2$$

(b) Show that, if the charges are all equal in magnitude and uniformly spaced around the circle, energy is radiated only into multiples of $N\omega_0$, but with an intensity N^2 times that for a single charge. Give a qualitative explanation of these facts.

(c) For the situatuon of part b, without detailed calculations show that for nonrelativistic motion the dependence on N of the total power radiated is dominantly as β^{2N}, so that in the limit $N \longrightarrow \infty$ no radiation is emitted.

(d) By arguments like those of part c show that for N relativistic particles of equal charge and symmetrically arrayed, the radiated power varies with N mainly as $e^{-\frac{2N}{3\gamma^3}}$ for $N \gg \gamma^3$, so that again in the limit $N \longrightarrow \infty$ no radiation is emitted.

(e) What relevance have the results of parts c and d to the radiation properties of a steady current in a loop?

Solution

1) The single particle radiation was derived from $\frac{d\rho_1}{d\Omega} = \frac{e^2}{4\pi c}|\widehat{n} \times (\widehat{n} \times \dot{\vec{\beta}})|^2$ this was also written $\frac{d\rho_1}{d\Omega} = const|A(t)|^2$ where $A(t) \propto$ E-field. If we consider an ensemble of particles, the E-fields from each particle are summed to give the total field so:

$$A_N(t) = \sum_{j=1}^{N} A_j(t)e^{\imath\phi_j}, \quad \text{with} \quad \phi_j = \omega t_j = m\omega_0 t_j \quad (\omega_0 t_j = \Theta_j)$$

thus,

$$\frac{d\rho_N}{d\Omega} = \frac{1}{4\pi c}|\sum_{j=1}^{N} q_j \widehat{n}_j \times (\widehat{n}_j \times \dot{\vec{\beta}}_j)e^{\imath m\Theta_j}|^2$$

on a circular orbit $\widehat{n}_j = \widehat{n} \quad \forall j$ and $\dot{\vec{\beta}}_j = \vec{\beta} \quad \forall j$ so:

$$\frac{d\rho_N}{d\Omega} = \frac{1}{4\pi c}|\widehat{n}_j \times (\widehat{n} \times \dot{\vec{\beta}})|^2|\sum_{j=1}^{N} q_j e^{\imath m\Theta_j}|^2 = \frac{d\rho_1(e=1)}{d\Omega} \times F(N)$$

2) If $q_j = q \quad \forall j$, then $F(N) = q^2|\sum_{j=1}^{N} e^{\imath m\Theta_j}|^2$

If particles are equally spaced on the circle then $\Theta_j = j\Delta\Theta$ with $\Delta\Theta = \frac{2\pi}{N}$ and so:

$$F(N) = q^2 |\sum_{j=1}^{N} e^{\imath mj\frac{2\pi}{N}}|^2$$

If $m = kN$ with $k \in \mathbb{N}$ then $F(N) = N^2 q^2$
So the radiation emitted at $m = kN$ is enhanced by a factor N^2 compare to a single-particle radiation.
3) We now consider the non-relativistic case, the second term in $fracd\rho_1(e=1)d\Omega$ dominates and thus

$$\frac{d\rho_N}{d\Omega} = \frac{e^2\omega_0^4 R^2}{2\pi c^3} m^2 [\frac{\cot^2\Theta}{\beta^2} J_m^2(m\beta\sin\Theta)]$$

$$\omega_0 = \frac{\beta c}{R}, \quad \text{and} J_m^2(m\beta\sin\Theta) = \frac{(m\beta\sin\Theta)^m}{2T(m+1)} \quad \text{give:}$$

$$\frac{d\rho_N}{d\Omega} = \frac{e^2\beta^4 c}{2\pi R^2}\frac{m^2}{\beta^2}\frac{m\beta\sin\Theta)^{2m}}{4T^2(m+1)}\cot^2\Theta$$

$$\propto \beta^{2(m+1)} \sim \beta^{2N}, \quad \text{when } m \approx N, n \gg 1$$

4) In the ultra-relativistic case and assuming $N \gg \gamma^3$ then

$$\omega \gg \omega_c \begin{cases} \omega = N\omega_0 = \frac{NR}{c} \\ \omega_c = \frac{3}{2}\gamma^3\frac{c}{R} \end{cases}$$

Therefore we can approximate the radiation spectrum by [1]:

$\frac{dI}{d\omega} \propto \sqrt{\frac{\omega}{\omega_c}} e^{-2\frac{\omega}{\omega_c}} \propto \left\{ e^{(-N\frac{c}{R})(\frac{2R}{3c\gamma^3})} \right.$

5) A steady current loop does not radiate as it circulates.

[1]This is valid for $\omega \gg \omega_c$