14.15

A particle of charge e moves in a circular path of radius R in the $x-y$ plane with a constant angular velocity ω_0.

(a) Show that the exact expression for the angular distribution of power radiated into the mth multiple of ω_0 is

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 R^2}{2\pi c^3} m^2 \{ \left[\frac{dJ_m(m\beta\sin\Theta)}{d(m\beta\sin\Theta)} \right]^2 + \frac{\cot^2\Theta}{\beta^2} J_m^2(m\beta\sin\Theta) \}$$

where $\beta = \frac{\omega_0 R}{C}$, and $J_m(x)$ is the Bessel function of order m.

(b) Assume nonrelativistic motion and obtain an approximate result for $\frac{dP_m}{d\Omega}$. Show that the results of Problem 14.4b are obtained in this limit.

(c) Assume extreme relativistic motion and obtain the results found in the text for a relativistic particle in instantaneously circular motion. [*Watson* (pp. 79, 249) may be of assistance to you.]

Solution

a) Consider the quantity A$_n$:

$$\vec{A_n} = -\frac{2\imath q n \omega_0^2}{(2\pi)^{\frac{3}{2}}\sqrt{2c}} \int_0^{\frac{2\pi}{\omega_0}} \hat{n} \times (\hat{n} \times \vec{\beta}) e^{\imath n\omega_0 (t' - \frac{\hat{n}\vec{r}}{c})}\, dt'$$

Defining the two polarizations vector $\varepsilon_\parallel, \varepsilon_\perp$; see Figure we have:

$$\widehat{\varepsilon_\parallel} = \widehat{x}; \qquad \widehat{n} = \cos\Theta\widehat{y} + \sin\Theta\widehat{z}$$
$$\widehat{\varepsilon_\perp} = \widehat{n} \times \widehat{n} = -\cos\Theta\widehat{z} + \sin\Theta\widehat{y}$$

so that

$$\widehat{n} \times (\widehat{n} \times \vec{\beta}) = \beta[\sin\omega_0 t\widehat{\varepsilon_\parallel} + \cos\omega_0 t\cos\Theta\widehat{\varepsilon_\perp}]$$

and the phase factor is:

$$\imath\omega(t' - \frac{\widehat{n}\vec{r}}{c}) = \omega[t' - \frac{r}{c}\sin\omega_0 t\sin\Theta]$$

therefore the above integral becomes (with the $\phi = \omega_0 t$)

$$I = \frac{\beta}{\omega_0}\int_0^{2\pi} \langle [-\sin\phi e^{\imath n(\phi - \frac{\omega_0 r}{c}\sin\phi\sin\Theta)}]\widehat{\varepsilon_\parallel}\, d\phi + \cos\Theta[\cos\phi e^{\imath n(\phi - \frac{\omega_0 r}{c}\sin\phi\sin\Theta)}]\widehat{\varepsilon_\perp}\, d\phi\rangle$$

$$= \frac{\beta}{\omega_0}\int_0^{2\pi} \langle [-\sin\phi e^{\imath n(\phi - x\sin\phi)}]\widehat{\varepsilon_\parallel}\, d\phi + \cos\Theta[\cos\phi e^{\imath n(\phi - x\sin\phi)}]\widehat{\varepsilon_\perp}\, d\phi\rangle$$

where $x \equiv \frac{\omega_0 r}{c}\sin\Theta$

We need to evaluate the two integrals:

$$I_1 = \int_0^{2\pi} -\sin\phi e^{\imath n(\phi - x\sin\phi)}\, d\phi, \quad \text{and}$$

$$I_2 = \int_0^{2\pi} \cos\phi e^{\imath n(\phi - x\sin\phi)}\, d\phi, \quad \text{where x was mentioned above}$$

Then the total integral will be: $I = \frac{\beta}{\omega_0}(I_1 + I_2)$

$$\text{Consider} \quad J_n(\tilde{x}) = \frac{-\imath^n}{2\pi} \int_0^{2\pi} e^{\imath(n\phi - \tilde{x}\sin\phi)}\, d\phi$$

then we see that:

$$\frac{d}{d\tilde{x}} J_n(\tilde{x}) = \frac{\imath^n}{2\pi} \int_0^{2\pi} e^{\imath(n\phi - \tilde{x}\sin\phi)} \sin\phi\, d\phi$$

with $\tilde{x} = nx$ we get:

$$(\widehat{\varepsilon_{\|}}) \qquad \frac{d}{d(nx)} J_n(nx) = \frac{\imath^n}{2\pi} I_1 \Leftrightarrow I_1 = -2\imath^{n+1}\pi \frac{d}{d(nx)} J_n(nx)$$

....

The second integral [1]

$$I_2 = \int_0^{2\pi} \cos\phi e^{\imath(n\phi - nx\sin\phi)}\, d\phi = \int_0^{2\pi} \frac{e^{\imath\phi} + e^{-\imath\phi}}{2} e^{\imath(n\phi - nx\sin\phi)}\, d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} (e^{\imath(n-1)\phi - \imath nx} + e^{\imath(n+1)\phi - \imath nx})\, d\phi = \frac{\imath}{4\pi} [J_{n+1}(nx) + J_{n-1}(nx)]$$

Using $J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$:

$$I_2 = \frac{2n\imath^n}{4\pi nx} J_n(nx) = \frac{n\imath^n}{2\pi nx} J_n(nx)$$

So putting I_1 and I_2 back in $\vec{A}$ and notify that the power angular distribution is $\frac{d\rho}{d\Omega} = |\vec{A}|$, one directly obtain given formula.

$$\frac{d\rho_m}{d\Omega} = \{[\frac{1}{T(m+1)} \frac{m}{2} (\frac{m\beta\sin\Theta}{2})^{m-1}]^2 + \frac{\cot^2\Theta}{\beta^2} [\frac{1}{T(m+1)} (\frac{m\beta\sin\Theta}{2})^m]^2\}$$

$$= \frac{e^2\omega_0^4 R^2 m^2}{2\pi c^3} \frac{1}{T^2(m+1)} \{(\frac{m}{2})(\frac{m}{2})^{2m-2} (\beta\sin\Theta)^{2m-2} + \frac{\cot^2\Theta}{\beta^2} (\frac{m}{2})^{2m} (\beta\sin\Theta)^{2m})\}$$

$$= \frac{e^2\omega_0^4 R^2 m^2}{2\pi c^3} (\frac{m}{2})^{2m} \frac{1}{T^2(m+1)} (\beta\sin\Theta)^{2m-2} [1 + \cos^2\Theta] \longrightarrow 0 \quad \text{when} \quad \beta \to 0 \quad \forall m \neq 1$$

So when $\beta \to 0$:

$$\frac{d\rho_m}{d\Omega} \simeq \frac{d\rho_1}{d\Omega} = \frac{e^2\omega_0^4 R^2}{8\pi c^3} (1 + \cos^2\Theta)$$

$\gamma \to \infty, \beta \simeq 1$

We want to show that $\frac{d\rho_m}{d\Omega}$ when $\beta \to 1$ gives equation 14.79.

Eq. (14.79) with $\gamma \to \infty$ gives:

$$\frac{d\rho}{d\Omega} = \frac{e^2}{3\pi^2 c} (\omega t)^4 [K^2_{\frac{2}{3}} (\frac{\omega^3 t^3}{3}) + K^2_{\frac{1}{3}} (\frac{\omega^3 t^3}{3})]$$

the relationship whit the limit $\beta = 1$ and $\gamma \to \infty$ for $\frac{d\rho_m}{d\Omega}$ is given by the Bessel identity:

$$J_m(x) \simeq \frac{1}{\pi} (\frac{2(m-x)}{3x})^{\frac{1}{2}} K_{\frac{1}{3}} (\frac{2^{\frac{3}{2}} (m-x)^{\frac{3}{2}}}{3x^{\frac{1}{2}}})$$

[1] Note we could have used the same trick for I_1, i.e. within $\sin\phi = \frac{1}{2\imath}(e^{\imath\phi} - e^{-\imath\phi})$