Prove, that

$$(\nabla_r^2 + k^2)G(\bar r) = 0$$

when $\bar r \neq \bar r'$

$$G_\pm = -\frac{1}{4\pi}\frac{e^{\pm\imath k|\bar r - \bar r'|}}{|\bar r - \bar r'|} = -\frac{1}{4\pi}\frac{e^{\pm\imath k\sqrt{r^2+r'^2-2rr'\cos\Theta}}}{\sqrt{r^2+r'^2-2rr'\cos\Theta}} \tag{1}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\Theta}\frac{\partial}{\partial\Theta}(\sin\Theta\frac{\partial f}{\partial\Theta}) + \frac{1}{r^2\sin\Theta}\frac{\partial^2 f}{\partial\phi^2} \tag{2}$$

Watch out [1] [2]

To simplify our notes one should use the following identity $r^2 + r'^2 - 2rr'\cos\Theta \equiv A$

$$\frac{\partial}{\partial r}\left(-\frac{r}{4\pi}\frac{e^{\pm\imath k\sqrt{A}}}{\sqrt{A}}\right) = -\frac{1}{4\pi}\left(\frac{e^{\pm\imath k\sqrt{A}}}{\sqrt{A}} \pm \frac{\imath kr(r-r'\cos\Theta)e^{\pm\imath k\sqrt{A}}}{A} - \frac{r(r-r'\cos\Theta)e^{\pm\imath k\sqrt{A}}}{A^{\frac{3}{2}}}\right)$$

$$\frac{\partial^2}{\partial r^2}(rG) = -\frac{1}{4\pi}\Bigg(\pm\frac{\imath k(r-r'\cos\Theta)e^{\pm\imath k\sqrt{A}}}{A} - \frac{r(r-r'\cos\Theta)e^{\pm\imath k\sqrt{A}}}{A^{\frac{3}{2}}}\pm$$

$$\pm\imath\frac{\left(\left(e^{\pm\imath k\sqrt{A}}kr + e^{\pm\imath k\sqrt{A}}k(r-r'\cos\Theta)\right) - \frac{\imath k^2 re^{\pm\imath k\sqrt{A}}(r-r'\cos\Theta)^2}{\sqrt{A}}\right)A - 2(r-r'\cos\Theta)^2 kre^{\pm\imath k\sqrt{A}}}{A^2} -$$

$$-\frac{\left(\left(e^{\pm\imath k\sqrt{A}}r + e^{\pm\imath k\sqrt{A}}(r-r'\cos\Theta)\right) - \frac{\imath kre^{\pm\imath k\sqrt{A}}(r-r'\cos\Theta)^2}{\sqrt{A}}\right)A^{\frac{3}{2}} - 3(r-r'\cos\Theta)^2 e^{\pm\imath k\sqrt{A}}A}{A^{\frac{5}{2}}}\Bigg)$$

$$\frac{\partial G}{\partial\Theta} = -\frac{1}{4\pi}\left(\frac{\pm\imath krr'\sin\Theta e^{\pm\imath k\sqrt{A}}}{A} - \frac{rr'\sin\Theta e^{\pm\imath k\sqrt{A}}}{A^{\frac{3}{2}}}\right)$$

$$\frac{\partial}{\partial\Theta}(\sin\Theta\frac{\partial G}{\partial\Theta}) = -\frac{1}{4\pi}\Bigg(\cos\Theta\left(\frac{\pm\imath krr'\sin\Theta e^{\pm\imath k\sqrt{A}}}{A} - \frac{rr'\sin\Theta e^{\pm\imath k\sqrt{A}}}{A^{\frac{3}{2}}}\right) + \sin\Theta\left(\frac{rr'\cos\Theta e^{\pm\imath k\sqrt{A}}}{A^{\frac{3}{2}}} - \frac{3r^2r'\sin^2\Theta e^{\pm\imath k\sqrt{A}}}{A^{\frac{5}{2}}}\right)\pm$$

$$\pm\imath\sin\Theta\left(\frac{krr'\cos\Theta e^{\pm\imath k\sqrt{A}}}{A} - \frac{kr^2r'^2\sin^2\Theta e^{\pm\imath k\sqrt{A}}}{A^2} - \frac{\imath k^2r^2r'^2\sin^2\Theta e^{\pm\imath k\sqrt{A}}}{A^{\frac{3}{2}}}\right)\Bigg)$$

$$\frac{1}{r}\frac{\partial^2}{\partial G^2} + \frac{1}{r^2\sin\Theta}\frac{\partial}{\partial\Theta}(\sin\Theta\frac{\partial G}{\partial\Theta}) = \frac{k^2}{4\pi}\frac{e^{\pm\imath k\sqrt{A}}}{\sqrt{A}}$$

$$\nabla^2 G + k^2 G = 0$$

Note, that $r \neq r'$

b)

$$\int\limits_V (\nabla_{\bar r}^2 + k^2)G(\bar r, \bar r')f(\bar r')\,d^3r' = f(r), \quad \text{V sphere of } \varepsilon \longrightarrow 0 \text{ around } r = r'$$

$$\underbrace{\int\limits_V \nabla\cdot(\nabla G)f(r')\,d^3r'}_{\text{according to Gauss's Theorem}} + k^2\int\limits_V G(r,r')f(r')\,d^3r' = \oint\limits_S \nabla Gf(r')\,da' + k^2\int\limits_V G(r,r')f(r')\,d^3r' == e^{\pm\imath k\varepsilon}f(r)$$

$$\lim_{\varepsilon\to 0(\equiv r=r')} e^{\pm\imath k\varepsilon}f(r) = f(r)$$

Since $f(\bar r)$ smooth and nonsingular, Taylor expansion always exists. So indeed $(\nabla_r^2 + k^2)G(r,r') = \delta^{(3)}(r-r')$ as it yields 0 everywhere except $r = r'$.

c)

$$\Psi^{(\pm)}(\bar r) = \underbrace{\phi(\bar r) + \frac{2m}{\hbar^2}\int G_\pm(\bar r, \bar r')V(\bar r')\Psi^{(\pm)}(\bar r')\,d^3r'}_{\text{inhomogenous Helmholtz equation}}, \quad \phi(\bar r) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{\imath\bar k\bar r}$$

$$\left[-\frac{\hbar^2}{2m}\nabla_{\bar r}^2 + V(\bar r)\right]\Psi^{(\pm)}(\bar r) = \frac{\hbar^2k^2}{2m}\Psi^{(\pm)}(\bar r) \Rightarrow V(\bar r)\Psi^{(\pm)}(\bar r) = \frac{\hbar^2}{2m}(\nabla_{\bar r}^2 + k^2)\Psi^{(\pm)}(\bar r) \Rightarrow$$

$$V(\bar r)\Psi^{(\pm)}(\bar r) = \frac{\hbar^2}{2m}(\nabla_{\bar r}^2 + k^2)\phi(\bar r)\int\underbrace{(\nabla_{\bar r}^2 + k^2)G_\pm(\bar r, \bar r')}_{\delta^{(3)}(\bar r - \bar r')}V(\bar r')\Psi^{(\pm)}(\bar r')\,d^3r' =$$

$$= \frac{\hbar^2}{2m}\left(\frac{1}{(2\pi)^{\frac{3}{2}}}(-k)^2e^{\imath\bar k\bar r} + k^2\frac{1}{(2\pi)^{\frac{3}{2}}}e^{\imath\bar k\bar r}\right) + V(\bar r)\Psi^{(\pm)}(\bar r) \quad \boxtimes$$

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[1] 1 Θ = angle between I and I'

[2] 2 Θ = azimuthal angle of I