Problem

(quantum field theory and the standard model, Matthew D. Schwartz (QFT MS), problem 17.1 , [1])

In supersymmetric theories for each fermion there is a scalar partner. For the electron e there is a selectron ($\tilde{e}$), muon μ - smuon $\tilde{\mu}$. For vector particle photon A_μ there is a fermion photino $\tilde{A}$.

Lagrangian should be something like (we took into account the misprint mentioned here).

$$\mathcal{L}_{SUSY} = \mathcal{L}_{SM} + \left(\partial_\mu \tilde{e} + igA_\mu \tilde{e}\right)\left(\partial_\mu \tilde{e}^* - igA_\mu \tilde{e}^*\right) + m_{\tilde{e}}^2 \tilde{e}^2 + g\tilde{e}\bar{e}\tilde{A} + \bar{\tilde{A}}\left(\partial\!\!\!/ + m_{\tilde{A}}\right)\tilde{A} + \\ + \left(\partial_\mu \tilde{\mu} + igA_\mu \tilde{\mu}\right)\left(\partial_\mu \tilde{\mu}^* - igA_\mu \tilde{\mu}^*\right) + m_{\tilde{\mu}}^2 \tilde{\mu}^2 + g\tilde{\mu}\bar{\mu}\tilde{A}. \tag{1}$$

Charge of smuon and selectron -1 (defined as g, $\alpha_e = \frac{g^2}{4\pi}$), g - coupling constant.

(a) Calculate magnetic moment for muon taking into account loop corrections from smuon presence.

Solution

Dirac equation with the presence of electromagnetic field has the form

$$\left(D\!\!\!\!/_\mu D\!\!\!\!/^\mu + m^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}\right)\psi = 0, \tag{2}$$

where $\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$, D_μ - gauge covariant derivative.

The contribution for muon magnetic moment will be coming only from terms with $\sigma^{\mu\nu}$.

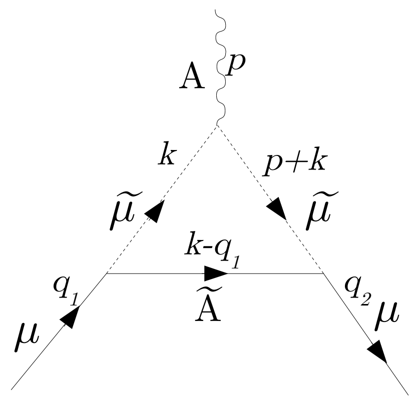

Figure 1: Photon radiation (A) by muon (μ) through the loop with smuon ($\tilde{\mu}$) and photino ($\tilde{A}$) with correspondent momenta q_1, k, $p = q_2 - q_1$, $p + k$, q_2. Muons with momenta q_1, q_2 are on-shell, while photon, smuon and photino are off-shell.

The contribution to anomalous magnetic moment is coming only from the diagram shown in figure 1. The contribution from diagrams with loops on muons or photons propagators are proportional to γ^μ, which are important only for renormalization of electron charge but not anomalous magnetic moment (equation (17.10) in [1]).

Hence, we need to calculate the contribution only from the diagram 1.

$$i\mathcal{M}_2^\mu = (-ig)^3 \int \frac{d^4k}{(2\pi)^4} \frac{i(k\!\!\!/ - q\!\!\!/_1 + m_{\tilde{A}})}{(k-q_1)^2 - m_{\tilde{A}}^2 + i\varepsilon} \bar{u}(q_2) \frac{i\left(k^\mu + (k^\mu + p^\mu)\right)}{(p+k)^2 - m_{\tilde{\mu}}^2 + i\varepsilon} \frac{i}{k^2 - m_{\tilde{\mu}}^2 + i\varepsilon} u(q_1). \tag{3}$$

To perform the integral (3) we will use the following relation

$$\frac{1}{ABC} = 2\int_0^1 \delta(x+y+z-1) \frac{1}{(Ax+By+Cz)^3} dxdydz. \tag{4}$$

Where in our case

$$\begin{aligned} A &= (k-q_1)^2 - m_{\tilde{A}}^2 + i\varepsilon, \\ B &= (p+k)^2 - m_{\tilde{\mu}}^2 + i\varepsilon, \\ C &= k^2 - m_{\tilde{\mu}}^2 + i\varepsilon. \end{aligned} \tag{5}$$

Taking into account (5) we can obtain

$$\begin{aligned} Ax + By + Cz = k^2 + xq_1^2 + yp^2 - m_{\tilde{\mu}}^2(y+z) - m_{\tilde{A}}^2 x + i\varepsilon + 2k(py - xq_1) = \\ = (k - q_1 x + yp)^2 + 2q_1 p x y + q_1^2 x(y+z) + p^2 y(x+z) - m_{\tilde{\mu}}^2(y+z) - m_{\tilde{A}}^2 x + i\varepsilon = \\ = (k - q_1 x + yp)^2 + xy(q_1+p)^2 + (q_1^2 x + p^2 y)z - m_{\tilde{\mu}}^2(y+z) - m_{\tilde{A}}^2 x + i\varepsilon = \\ = (k - q_1 x + yp)^2 + xym^2 + m^2 xz + p^2 yz - m_{\tilde{\mu}}^2(y+z) - m_{\tilde{A}}^2 x + i\varepsilon = \\ = (k - q_1 x + yp)^2 + xm^2(1-x) + p^2 yz - m_{\tilde{\mu}}^2(1-x) - m_{\tilde{A}}^2 x + i\varepsilon = \\ = (k - q_1 x + py)^2 - \Delta + i\varepsilon, \end{aligned} \tag{6}$$

where we used $x+y+z=1$ (from Dirac delta function) and we used that muons with momenta q_1, q_2 are on-shell: $q_1^2 = q_2^2 = m^2$, $\Delta = m_{\tilde{A}}^2 x + m_{\tilde{\mu}}^2(1-x) - m^2 x(1-x) - p^2 yz$.

Let's make a shift $k^\mu \to k^\mu + q_1^\mu x - p^\mu y$, which leaves d^4k the same.

We expand numerator of (4)

$$\begin{aligned} N^\mu = \bar{u}(q_2)\left(\not{k} - \not{q}_1 + m_{\tilde{A}}\right)(2k^\mu + p^\mu)\, u(q_1) = \bar{u}(q_2) 2k^\mu \left(\left(k^\nu - q_1^\nu\right)\gamma^\nu + m_{\tilde{A}}\right) u(q_1) + \\ + \bar{u}(q_2) p^\mu \left(\left(k^\nu - q_1^\nu\right)\gamma^\nu + m_{\tilde{A}}\right) u(q_1) \to [\text{momentum shift } k^\mu] \to \\ \to \bar{u}(q_2) 2\left(k^\mu + q_1^\mu x - p^\mu y\right)\left(\left(k^\nu + q_1^\nu(x-1) - p^\nu y\right)\gamma^\nu + m_{\tilde{A}}\right) u(q_1)) \\ + \bar{u}(q_2) p^\mu \left(\left(k^\nu + q_1^\nu(x-1) - p^\nu y\right)\gamma^\nu + m_{\tilde{A}}\right) u(q_1)) = \\ = \bar{u}(q_2) 2\left(k^\mu + \frac{Q^\mu}{2}x - p^\mu(y + x/2)\right)\left(\left(k^\nu + \frac{Q^\nu}{2}(x-1) - p^\nu\left(y + \frac{x-1}{2}\right)\right)\gamma^\nu + m_{\tilde{A}}\right) u(q_1)) + \\ + \bar{u}(q_2) p^\mu \left(\left(k^\nu + \frac{Q^\nu}{2}(x-1) - p^\nu\left(y + \frac{x-1}{2}\right)\right)\gamma^\nu + m_{\tilde{A}}\right) u(q_1)), \end{aligned} \tag{7}$$

where $Q^\mu = q_2^\mu + q_1^\mu$ (taking into account that $p^\mu = q_2^\mu - q_1^\mu$).

To simplify the expression we use that $k^\mu k^\nu = \frac{1}{4} g^{\mu\nu} k^2$, and that all linear terms on k^μ are "0" after integration over d^4k. Hence, one can derive

$$\begin{aligned} N^\mu = \bar{u}(q_2)\left(\frac{k^2}{2}\gamma^\mu + \frac{Q^\mu}{2}x(x-1)\not{Q} + p^\mu(y + x/2)(y-z)\not{p} - \frac{Q^\mu}{2}x(y-z)\not{p} - p^\mu(y+x/2)(x-1)\not{Q} + \right. \\ \left. + \frac{p^\mu}{2}(x-1)\not{Q} - \frac{p^\mu}{2}(y-z)\not{p} + m_{\tilde{A}}\left(Q^\mu x - p^\mu(y-z)\right)\right) u(q_1) = \\ = \bar{u}(q_2)\left(Q^\mu\left(x(x-1)m + x m_{\tilde{A}}\right) + p^\mu(z-y)\left((x-1)m + m_{\tilde{A}}\right) + \gamma^\mu \frac{k^2}{2}\right) u(q_1), \end{aligned} \tag{8}$$

where we used that $\not{q}_1 u(q_1) = m u(q_1)$, $\bar{u}(q_2)\,\not{q}_2 = \bar{u}(q_2)\, m$ (from which we have $\bar{u}(q_2)\,\not{Q}\, u(q_1) = \bar{u}(q_2)\, 2m\, u(q_1)$ and $\bar{u}(q_2)\,\not{p}\, u(q_1) = 0$).

Let's plug all terms in initial integral (3)

$$\begin{aligned} i\mathcal{M}_2^\mu = 2\int_0^1 dx dy dz\, \delta(x+y+z-1)\delta(x+y+z-1) \times \\ g^3 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(q_2)\left(Q^\mu x\left((x-1)m + m_{\tilde{A}}\right) + p^\mu(z-y)\left((x-1)m + m_{\tilde{A}}\right) + \gamma^\mu \frac{k^2}{2}\right) u(q_1)}{\left(k^2 - \Delta + i\varepsilon\right)^3}. \end{aligned} \tag{9}$$

It is seen that the term with p^μ is anti-symmetric for $y \longleftrightarrow z$, while the rest of the integral is symmetric. Hence, the term proportional to p^μ after integration over (x, y) is zero.

Let's use Gordon identity for the term with Q^μ

$$\bar{u}(q_2)Q^\mu u(q_1) = 2m\bar{u}(q_2)\gamma^\mu u(q_1) - i\bar{u}(q_2)\sigma^{\mu\nu}p_\nu u(q_1) \tag{10}$$

and we obtain the following integral

$$\begin{aligned} i\mathcal{M}_2^\mu = 2\int_0^1 dxdydz\,\delta(x+y+z-1)\delta(x+y+z-1)\times \\ g^3\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(q_2)\left(\gamma^\mu\left(\frac{k^2}{2}+x(x-1)m+xm_{\tilde{A}}\right)-i\sigma^{\mu\nu}p_\nu x\left((x-1)m+m_{\tilde{A}}\right)\right)u(q_1)}{\left(k^2-\Delta+i\varepsilon\right)^3}. \end{aligned} \tag{11}$$

The expression (13) satisfies Ward identity (as it was anticipated)

$$p_\mu\mathcal{M}^\mu = 0. \tag{12}$$

As it was mentioned above, the contribution to anomalous magnetic moment will come from the term proportional to $\sigma^{\mu\nu}$

$$i\mathcal{M}_2 = \bar{u}(q_2)\left(\frac{e}{2m}F_2\sigma^{\mu\nu}p_\nu + \ldots \text{остальные слагаемые}\right)u(q_1). \tag{13}$$

And F_2 has the form

$$F_2 = -\frac{2m}{g}2ig^3\int_0^1 dxdydz\,\delta(x+y+z-1)x\left((x-1)m+m_{\tilde{A}}\right)\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left(k^2-\Delta+i\varepsilon\right)^3}. \tag{14}$$

integration over "k" can be carried out due to Wick rotation $k_0 \to ik_0$ ($k^2 \to -k_0^2 - \mathbf{k}^2 = -k_E^2$). ε guarantees that the integral doesn't diverge. After computation of the integral we can set $\varepsilon = 0$. Transforming the integral (14) to spherical system of coordinate, one obtains

$$-i\int_0^\infty\frac{dk_E}{(2\pi)^4}\frac{2\pi^2k_E^3}{\left(k_E^2+\Delta\right)^3} = -\frac{i}{32\pi^2\Delta}. \tag{15}$$

Substituting (15) in (14) we have

$$\begin{aligned} F_2 = -\frac{4g^2m}{32\pi^2}\int_0^1\frac{\delta(x+y+z-1)x\left((x-1)m+m_{\tilde{A}}\right)}{m_{\tilde{A}}^2x+m_{\tilde{\mu}}^2(1-x)-m^2x(1-x)-p^2yz}dxdydz = \\ -\frac{g^2m}{8\pi^2}\int_0^1\int_0^{1-x}\frac{x\left((x-1)m+m_{\tilde{A}}\right)}{m_{\tilde{A}}^2x+m_{\tilde{\mu}}^2(1-x)-m^2x(1-x)-p^2y(1-x-y)}dydx. \end{aligned} \tag{16}$$

For $p = 0$ (photon energy is much smaller than mass of muon, smuon and photino) the integral has an analytic form

$$F_{2(p=0)} = -\frac{g^2}{8\pi^2}\int_0^1\frac{(1-x)x\left((x-1)+\epsilon_{\tilde{A}}\right)}{x^2-x(1-\epsilon_{\tilde{A}}^2)+\epsilon_{\tilde{\mu}}^2(1-x)}dx, \tag{17}$$

where $\epsilon_{\tilde{A}} = \frac{m_{\tilde{A}}}{m}$, $\epsilon_{\tilde{\mu}} = \frac{m_{\tilde{\mu}}}{m}$.

In the limit when muon mass is much smaller than mass of smuon and photino ($\epsilon_{\tilde{\mu}} >> 1$, $\epsilon_{\tilde{A}} >> 1$)

$$F_{2(p=0)} \approx -\frac{g^2}{8\pi^2}\int_0^1\frac{(1-x)x\epsilon_{\tilde{A}}}{x\epsilon_{\tilde{A}}^2+\epsilon_{\tilde{\mu}}^2(1-x)}dx = -\frac{g^2}{16\pi^2}\frac{\epsilon_{\tilde{A}}\left(-\epsilon_{\tilde{A}}^4+\epsilon_{\tilde{A}}^2\epsilon_{\tilde{\mu}}^2\log\left(\frac{\epsilon_{\tilde{A}}^4}{\epsilon_{\tilde{\mu}}^4}\right)+\epsilon_{\tilde{\mu}}^4\right)}{(\epsilon_{\tilde{A}}-\epsilon_{\tilde{\mu}})^3}. \tag{18}$$

For $m_{\tilde{\mu}} >> m_{\tilde{A}}$ we obtain

$$F_{2(p=0)} \approx \frac{g^2}{16\pi^2}\frac{m_{\tilde{A}}^2}{m^2}, \tag{19}$$

for $m_{\tilde{A}} >> m_{\tilde{\mu}}$ we have

$$F_{2(p=0)} \approx -\frac{g^2}{16\pi^2}\frac{m_{\tilde{\mu}}^2}{m_{\tilde{A}}^2}, \tag{20}$$

and when $m_{\tilde{\mu}} \approx m_{\tilde{A}}$

$$F_{2(p=0)} \approx -\frac{g^2}{48\pi^2}\frac{m_{\tilde{A}}}{m_{\tilde{A}}}. \tag{21}$$

There were some similar estimations in [2, 3, 4] but it doesn't look the same...

References

[1] *Quantum Field Theory and the Standard Model*, Matthew D. Schwartz, Cambridge University Press (2014-03-06), 863p.

[2] *The Second Order Weak Correction to (G-2) of the Muon in Arbitrary Gauge Models*, Jacques P. Leveille, Nucl.Phys. B137 (1978) 63-76 (1978).

[3] *Constraints on Supersymmetric Particle Masses From (g−2)* μ, J.A. Grifols, A. Mendez, Phys.Rev. D26 (1982) 1809.

[4] *Limits on supersymmetry masses from the anomalous magnetic moment of leptons*, O.F. Syljuasen, H.A. Olsen, Phys.Scripta 48 (1993) 525-526.