12.6

A particle of mass m and charge e moves in the laboratory in crossed, static, uniform, electric and magnetic fields. **E** is parallel to the x axis; **B** is parallel to the y axis.

(a) For $|\mathbf{E}| < |\mathbf{B}|$ make the necessary Lorentz transformation described in Section 12.3 to obtain explicitly parametric equations for the particle's trajectory.

(b) Repeat the calculation of part a for $|\mathbf{E}| > |\mathbf{B}|$.

Solution

Part (a) can be done from the results from part (b)...
So let's first do (b):
(b) the covariant equation of motion is:

$$\frac{du^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} u_\beta$$

F: field-strength tensor and u is 4-velocity

if $\vec{E} \parallel \vec{B} \parallel \widehat{z}$ then

$$F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & -B_z & 0 \\ 0 & B_z & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix}$$

so the 4-equations for the 4-velocity components are:

$$\begin{cases} \frac{du^0}{d\tau} &= -\frac{e}{mc} E_z u_3 = \frac{e}{mc} E_z u^3 \\ \frac{du^1}{d\tau} &= -\frac{e}{mc} B_z u_2 = \frac{e}{mc} B_z u^2 \\ \frac{du^2}{d\tau} &= \frac{e}{mc} B_z u_1 = -\frac{e}{mc} B_z u^1 \\ \frac{du^3}{d\tau} &= \frac{e}{mc} E_z u_0 = \frac{e}{mc} E_z u^0 \end{cases}$$

That is two sets of couple ODE:

$$(1) \begin{cases} \frac{du^0}{d\tau} &= \frac{e}{mc} E_z u^3 \\ \frac{du^2}{d\tau} &= -\frac{e}{mc} B_z u^1 \end{cases}, \qquad (2) \begin{cases} \frac{du^1}{d\tau} &= \frac{e}{mc} B_z u^2 \\ \frac{du^2}{d\tau} &= -\frac{e}{mc} B_z u^1 \end{cases}$$

set (1) gives:

$$\frac{d^2 u^0}{d\tau^2} = \frac{e}{mc} E_z \frac{du^3}{d\tau} = (\frac{e}{mc} E_z)^2 u^0$$

and $u^0 = C_1 \cosh(\kappa\tau), u^3 = C_1 \sinh(\kappa\tau)$ where $\kappa = \frac{e}{mc} E_z$ (3)

set (2) was solved in class: let $S = u^1 + \imath u^2$ then set (2) can be casted in one ODE:

$$\frac{dS}{d\tau} = -\frac{\imath e B}{mc} S \Rightarrow S = C_2 e^{-\frac{\imath e B}{mc}\tau}$$

$\Rightarrow u^1 = C_2 \cos(\frac{eB}{mc}\tau), \quad u^2 = -C_2 \sin(\frac{eB}{mc}\tau)$ (4)

Let's follow JDJ and introduce $\rho \equiv \frac{E}{B}$ and $\phi = \frac{eB}{mc}\tau$.

Then we got: $u^0 = C_1 \cosh(\phi\rho), \quad u^3 = C_1 \sinh(\phi\rho); \quad u^1 = C_2 \cos(\phi), \quad u^2 = C_2 \sin(\phi)$.

The norm of the 4-velocity is $u^\alpha u_\alpha = c^2$, this gives a condition between constants C_1 and C_2:

$$u^\alpha u_\alpha = C_1^2[\cosh^2(\phi\rho) - \sinh^2(\phi\rho)] - C_2^2[\cos^2(\phi) + \sin^2(\phi)] = c^2$$

$$\Rightarrow C_1^2 - C_2^2 = c^2$$

as suggested in (4) take $C_2 = Ac$, then $C_1 = c\sqrt{1+A^2}$ and
$u^0 = C\sqrt{1+A^2}\cosh(\phi\rho), \quad u^3 = C\sqrt{1+A^2}\sinh(\phi\rho); \quad u^1 = Ac\cos(\phi), \quad u^2 = Ac\sin(\phi)$.

integrate w.r.t. time to get equation of motion:

$$\begin{cases} ct = c\sqrt{1+A^2}\frac{mc}{EB}\frac{B}{E}\sinh(\phi\rho) \equiv \frac{R}{\rho}\sqrt{1+A^2}\sinh(\phi) \\ z = \frac{R}{\rho}\sqrt{1+A^2}\cosh(\phi\rho) \end{cases}$$

and

$$\begin{cases} x = AR\sin(\phi\rho) \\ y = AR\cos(\phi\rho) \end{cases}, \text{ where } R \equiv \frac{mc^2}{EB}$$

(a) case where $\vec{E}$ and $\vec{B}$ make an angle Θ. Consider $\vec{E} = E\hat{z}$, $\vec{B} \in (x-z)$ plane. $\Theta = \angle(\vec{B}, \hat{z})$ in referential K. We need to find a Lorentz transformation such that in x', $\vec{B} \parallel \vec{E}$. Consider $\vec{\beta} = \beta\hat{y}$ in K then $\vec{\beta}\vec{E} = 0$, $\vec{\beta}\vec{\beta} = 0$, $\vec{\beta} \times \vec{E} = \beta E\hat{x}$ and $\vec{\beta} \times \vec{B} = \beta B(\cos\Theta\hat{x} - \sin\Theta\hat{z})$.

Using JDJ Eq. 11.144 in K′:

$$\vec{E'} = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) = \gamma[E\hat{z} + \beta B\cos\Theta\hat{x} - \beta B\sin\Theta\hat{z}] = \gamma[(E - \beta B\sin\Theta)\hat{z} + \beta B\cos\Theta\hat{x}]$$
$$\vec{B'} = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) = \gamma[B\sin\Theta\hat{x} + B\cos\Theta\hat{z} - \beta E\hat{x}] = \gamma[B\cos\Theta\hat{z} + (B\sin\Theta - \beta E)\hat{x}]$$

$$\vec{E'} \times \vec{B} = 0 \Leftrightarrow \beta B^2\cos^2\Theta - (E - \beta B\sin\Theta)(B\sin\Theta - \beta E) = 0$$
$$\beta B^2\cos^2\Theta - (EB\sin\Theta - \beta E^2 - \beta b^2\sin^2\Theta + \beta^2 BE\sin\Theta) = 0$$
$$\beta B^2 + \beta E^2 - EB\sin\Theta(1+\beta^2) = 0$$

So β should verify:

$$\frac{\beta}{1+\beta^2} = \frac{EB\sin\Theta}{E^2+B^2} = \frac{\vec{E}\times\vec{B}}{E^2+B^2}$$

In the frame K′, $\vec{B'}$ and $vecE'$ are parralel both make an angle Ψ with z′. The angle Ψ can be found from $\vec{B'}$; we have $\vec{B'} = B_0(\cos\Psi\hat{z} + \sin\Psi\hat{x})$ in the new frame ($\vec{B} \perp \vec{\beta}$).

So $\tan\Psi = \frac{B\sin\Theta - \beta E}{B\cos\Theta}$.

So to have the same case as in (b) $\vec{B} \parallel \vec{E} \parallel \hat{z}$ we need to do a rotation of K′ by $\Psi \to$ K″. In K ″ the equation of motion are given by (b) and one "just" need to do the inverse rotation and Lorentz boost to obtain the general equation of motion in K.