Models of nonlinear acoustics

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Hydrodynamics

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water which it had put in motion; it rolled forward with great velocity, assuming the form of a large solitary elevation which continued its course along the channel apparently without change of form or diminution of speed...

Figure: John Scott Russell [Courtesy of Wikipedia]

Experiments with water solitons

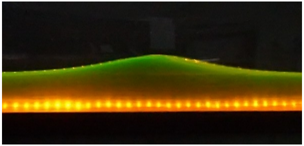


Figure: Soliton [Courtesy of Wikipedia]

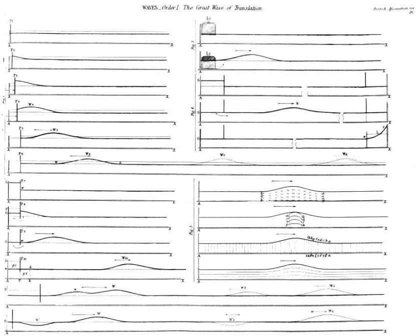


Figure: Wave tank [Courtesy of Chris Eilbeck/Heriot-Watt University]

Main idea of nonlinear dynamics - we cannot obtain this effects by a linear approach. Superposition principle fails.

Acoustics

"The wave of translation in the oceans of water, air, and ether" (1885)

- description of the experiment, where observers heard the report of the gun first, while the order to fire reached them second.

Conclusion

Disturbance with the high amplitude propagates faster than the other one with the low.

Further investigations

- Korteweg - De Vries (KdV) equation [Korteweg and De Vries, 1895]:

$$\partial_t \phi + \partial_t^3 \phi + 6\phi \partial_x \phi = 0$$

  - describes Russel's solitons
- Fermi - Pasta - Ulam (FPU) problem (1955) - nonlinear system exhibit complex behavior
- connection between FPU problem and KdV equation - [Zabusky and Kruskal, 1965]
- first nonlinear acoustic equation [Kuznetsov, 1971]:

$$\partial_t^2 \widetilde{\phi} - \Delta \widetilde{\phi} = \epsilon \partial_t [b \Delta \widetilde{\phi} + (\nabla \widetilde{\phi})^2 + a(\partial_t \widetilde{\phi})^2]$$

Main results for the KZK equation: $2\partial^2_{\tau\xi_1}\rho_1 - \partial^2_{\xi_2}\rho_1 = b\partial^3_\tau\rho_1 + a'\partial^2_\tau\rho_1^2$

Main Results in this area [by A. Rozanova-Pierrat]

- Local unique solvability $\forall u_0 \in H^s, s > [\frac{n}{2}] + 1$;
- For viscous case $(\beta > 0)\exists!$ global solution for sufficiently small initial data;
- For the case $\beta = 0 \quad \nexists$ global in time smooth solution (i.e. $\exists$ shock wave);
- This equation is indeed the approximation of NS system for some finite time.

Canonical form of Navier-Stokes system/Equation of state

$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, & \text{(1a)} \\
\rho\left(\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u}\right) = -\nabla p + (\frac{1}{3}\eta + \xi)\nabla\nabla \cdot \vec{u} + \eta \Delta \vec{u}, & \text{(1b)} \\
\rho T(\partial_t s + \vec{u} \cdot \nabla s) = \frac{\eta}{2}\left[\partial_k u_i + \partial_i u_k - \frac{2}{3}\delta_{ik}\partial_l u_l\right]^2 + \xi(\nabla \cdot \vec{u})^2 + \kappa \Delta T, & \text{(1c)}
\end{cases}
$$

Equation of state for an ideal gas

$$p(\rho, T) = \rho R T \Rightarrow T(p, \rho) = \frac{p}{\rho R}$$

Assumptions for all the equations

Order of approximation

- $p(x,y,z,t) = p_0 + \epsilon p_1(x,y,z,t) + \epsilon^2 p_2(x,y,z,t)$; we work in $\mathbb{R}^3$
- $\rho(x,y,z,t) = \rho_0 + \epsilon \rho_1(x,y,z,t)$
- $\vec{u}(x,y,z,t) = 0 + \epsilon \vec{v}(x,y,z,t)$
- $T(x,y,z,t) = T_0 + \epsilon T_1(x,y,z,t)$
- viscosity and heat conduction number are also small (the same order): $\frac{4}{3}\eta + \xi = \epsilon\delta$, $\kappa = \epsilon\bar{\kappa}$
- for the work we have to take into account every term up to second-order
- $\epsilon = \frac{|\vec{u}|}{c_0}$

Assumption of irrotational flow

$$\nabla \times \vec{u} = 0 \Rightarrow \vec{u} = -\nabla\phi$$

- For simplicity let's use $c_0 = 1$

Kuznetsov equation

Derivation of Kuznetsov equation: $\partial_t^2\phi - \Delta\phi = \epsilon\partial_t[b\Delta\phi + (\nabla\phi)^2 + a(\partial_t\phi)^2]$

Original work – [Kuznetsov, 1971]

The law of mass conservation

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

$$\Downarrow$$

After expansion

$$\epsilon(\partial_t \rho_1 + \rho_0 \nabla \cdot \vec{v}) + \epsilon^2(\nabla \cdot (\rho_1 \vec{v})) = 0$$

Linear theory: $\partial_t \rho_1 + \rho_0 \nabla \cdot \vec{v} = 0$

Momentum conservation law

$$\rho\,(\partial_t\vec{u}+(\vec{u}\cdot\nabla)\vec{u})=-\nabla p+(\frac{1}{3}\eta+\xi)\nabla\nabla\cdot\vec{u}+\eta\Delta\vec{u}$$

$$\Downarrow$$

Differential relations after expansion

$$\epsilon(\rho_0\partial_t\vec{v}+\nabla\rho_1)+\epsilon^2(\rho_0(\vec{v}\cdot\nabla)v+\rho_1\partial_t\vec{v}+\frac{\gamma-1}{2\rho_0}\nabla\rho_1^2+\frac{\rho_0}{\gamma C_V}\nabla s_1-\delta\Delta\vec{v})=0$$

Linear theory: $\rho_0\partial_t\vec{v}+\nabla\rho_1=0$

Energy conservation law

$$\rho T(\partial_t s + \vec{u}\cdot\nabla s) = \frac{\eta}{2}\left[\partial_k u_i + \partial_i u_k - \frac{2}{3}\delta_{ik}\partial_l u_l\right]^2 + \xi(\nabla\cdot\vec{u})^2 + \kappa\Delta T$$

and

Equation of state:

$$p = \rho R T$$

$$\Downarrow$$

Entropy (leading term)

Second order only: $\epsilon^2(\partial_t s_1) = \epsilon^2(\frac{\bar{\kappa}(\gamma-1)}{\rho_0^2}\Delta\rho_1)$

Final form of the equation

So in this way we have:

$$\partial_t^2 \widetilde{\phi} - \Delta \widetilde{\phi} = \epsilon \partial_t [\underbrace{b \Delta \widetilde{\phi}}_{\text{dissipation}} + \underbrace{(\nabla \widetilde{\phi})^2}_{\text{local nonlinear effects}} + \underbrace{a(\partial_t \widetilde{\phi})^2]}_{\text{global nonlinear effects}} + O(\epsilon^2)$$

where

$$a \equiv \frac{\gamma - 1}{2}, \qquad b \equiv \frac{1}{\rho_0}[\delta + \bar{\kappa}\left(\frac{1}{C_V} - \frac{1}{C_P}\right)]$$

Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation

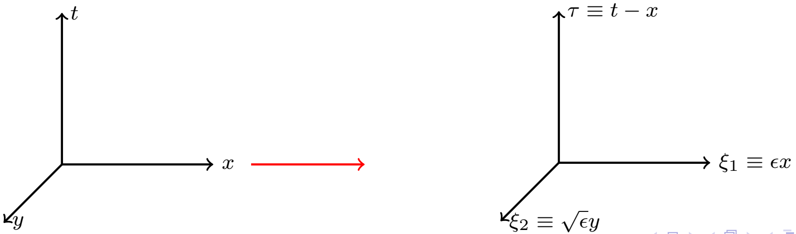
KZK equation: $2\partial^2_{\tau\xi_1}\rho_1 - \partial^2_{\xi_2}\rho_1 = [b\partial^3_\tau\rho_1 + a'\partial^2_\tau\rho_1^2]$

Original works

[Zabolotskaya and Khokhlov, 1969] and [Kuznetsov, 1971]

The idea of derivation

- small disturbance of the NS system
- paraxial approximation
  - the solution *a priori* has a preferred direction of propagation
  - slowly varies in longitudinal direction
  - slowly in the transversal direction



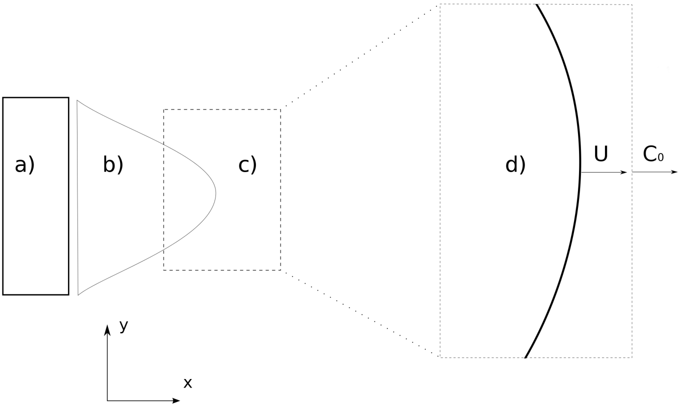


Figure: Scheme of coordinate transform: a) a source of sound; b) directional sound beam; c) the tip of this beam; d) *frame*, that moves with the speed of sound c_0, and the rescaled tip (approximately plane wave), that moves with his own velocity $\vec{u}$

Derivation of the KZK equation: $2\partial^2_{\tau\xi_1}\rho_1 - \partial^2_{\xi_2}\rho_1 = [b\partial^3_\tau\rho_1 + a'\partial^2_\tau\rho_1^2]$

Main ways

- directly from NS system
- from the Kuznetsov equation

Further steps

- Differentiation with respect to t
- Change of variables, as described above

Result of operations

$$\langle 2\partial^2_{\tau\xi_1} - \partial^2_{\xi_2}\rangle \rho_1 = [b\partial^3_\tau \rho_1 + a'\partial^2_\tau \rho_1^2] + O(\epsilon)$$

where

$$a' = \frac{1+a}{\rho_0}$$

and

$$\rho_1 = \rho_0 \partial_t \widetilde{\phi}$$

Examples of numerical simulations and fields of application

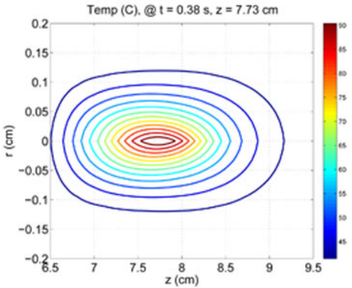


Figure: HIFU-modeling [Courtesy of Sysoneson]

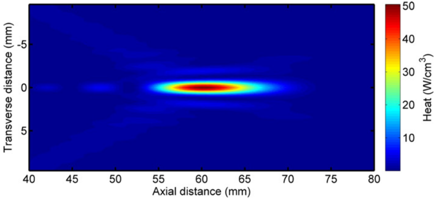


Figure: Heating profile produced by the KZK simulation of the 1.06 MHz spherically focused transducer at 6.3 MPa [Courtesy of Jensen, Cleveland and Coussios]

- to palpate tissue locally within the focal region
- stone communition with lithotripsy
- high-intensity focused ultrasound (to heat and ablate tissue)

Nonlinear progressive wave (NPE) equation

NPE: $-2\partial^2_{\bar{\tau}\bar{\xi}_1} R - \partial^2_{\bar{\xi}_2} R = -b\partial^3_{\bar{\xi}_1} R + (a+1)\partial^2_{\bar{\xi}_1} R^2$

Main step

The idea is the same (rescaling, which depends on the main parameter of system - ϵ) :

$$\begin{cases} t \to \epsilon t = \bar{\tau}, \\ x \to x - t = \bar{\xi}_1, \\ y \to \sqrt{\epsilon} y = \bar{\xi}_2 \end{cases}$$

Derivation of the NPE: $-2\partial^2_{\bar{\tau}\bar{\xi}_1} R - \partial^2_{\bar{\xi}_2} R = -b\partial^3_{\bar{\xi}_1} R + (a+1)\partial^2_{\bar{\xi}_1} R^2$

Main ways

- directly from NS system [McDonald, 1987]
- from the Kuznetsov equation [we checked, that this idea is valid as in the case of KZK equation]

Simplest way to derive

We already have a perturbed NS system in the form of the Kuznetsov equation

$$\partial_t^2 \widetilde{\phi} - \Delta \widetilde{\phi} = \epsilon \partial_t [b \Delta \widetilde{\phi} + (\nabla \widetilde{\phi})^2 + a(\partial_t \widetilde{\phi})^2] \tag{2}$$

Differentiation and expansion

$$\partial_t[(2)] \rightarrow \partial_t^3 \widetilde{\phi} - \Delta \partial_t \widetilde{\phi} = \epsilon \partial_t^2 [b \Delta \widetilde{\phi} + (\nabla \widetilde{\phi})^2 + a(\partial_t \widetilde{\phi})^2]$$

$$2\partial^2_{\bar{\tau}\bar{\xi}_1}(\partial_{\bar{\xi}_1}\widetilde{\phi}) + \partial^2_{\bar{\xi}_2}(\partial_{\bar{\xi}_1}\widetilde{\phi}) = b\partial^3_{\bar{\xi}_1}(\partial_{\bar{\xi}_1}\widetilde{\phi}) + (a+1)\partial^2_{\bar{\xi}_1}(\partial_{\bar{\xi}_1}\widetilde{\phi})^2 + O(\epsilon^2)$$

Implicit form

Dependence between ρ_1 and $\widetilde{\phi}$ from the previous calculation

$$\rho_1 = \rho_0 \partial_t \widetilde{\phi}$$

In framed coordinate system

$$\rho_1 = -\rho_0 \partial_{\bar{\xi}_1} \widetilde{\phi} + \epsilon \rho_0 \partial_{\bar{\tau}} \widetilde{\phi}$$

For the dominating order only!

$$\partial_{\bar{\xi}_1} \widetilde{\phi} = -\frac{\rho_1}{\rho_0} \equiv -R$$

Full form of NPE equation

$$-2\partial^2_{\bar{\tau}\bar{\xi}_1} R - \partial^2_{\bar{\xi}_2} R = -b\partial^3_{\bar{\xi}_1} R + (a+1)\partial^2_{\bar{\xi}_1} R^2$$

Comparison of two equations

Visual comparison

KZK

$$\partial^2_{\tau\xi_1}\rho_1 - \partial^2_{\xi_2}\rho_1 = [b\partial^3_\tau\rho_1 + a'\partial^2_\tau\rho_1^2]$$

NPE

$$-2\partial^2_{\bar{\tau}\bar{\xi}_1}R - \partial^2_{\bar{\xi}_2}R = -b\partial^3_{\bar{\xi}_1}R + (a+1)\partial^2_{\bar{\xi}_1}R^2$$

(!) Direct attempt to change roles of longitudinal coordinate and time fails.

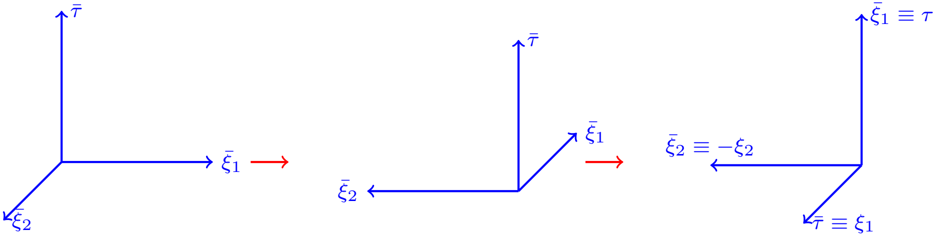
We remind NPE equation in terms of potential $\widetilde{\phi}$

$$2\partial^2_{\bar{\tau}\bar{\xi}_1}(\partial_{\bar{\xi}_1}\widetilde{\phi}) + \partial^2_{\bar{\xi}_2}(\partial_{\bar{\xi}_1}\widetilde{\phi}) = b\partial^3_{\bar{\xi}_1}(\partial_{\bar{\xi}_1}\widetilde{\phi}) + (a+1)\partial^2_{\bar{\xi}_1}(\partial_{\bar{\xi}_1}\widetilde{\phi})^2$$

and

$$\partial_{\bar{\xi}_1}\widetilde{\phi} = -\frac{\rho_1}{\rho_0} \equiv -R$$

Simple coordinate transform



So, we've kept our coordinate system right-handed and at the same time we've changed all signs in the NPE equation in such way that we have now a pure bijection.

Applications of NPE

One shall say briefly about modern area of applications of NPE. Despite the fact, that there are not so much analytical studies of the equation, it has a wide spectrum of applications:

- Locey in his work used NPE equation to model atmosphere turbulence effects [Locey, [2008]]. Because of the fact, that turbulence influences on the index of refraction of atmosphere, now this fluctuations can be corrected.
- NPE is widely used to model sonic boom/blast waves propagation in different medium, mostly near a ground surface in the air and water [Leissing et al. [2008], Leissing [2007, 2009], McDonald [2009], Piacsek and Plotkin [2013], van der Eerden and van den Berg]
- to model nonlinear sound wave propagation in the stratified atmospheres [Edward McDonald and Piacsek [2011]]

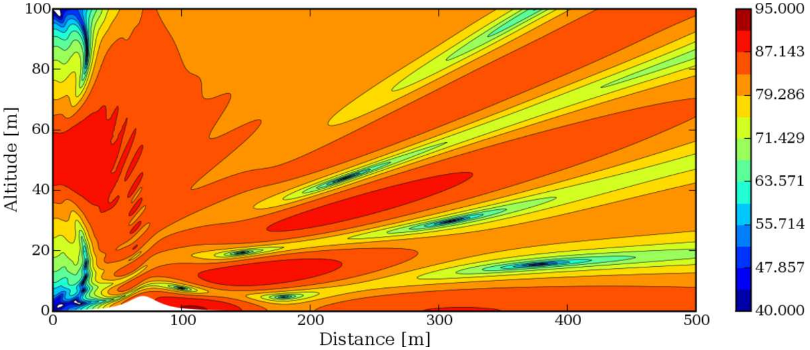


Figure: Sound Pressure Level map calculated with the NPE[Courtesy of Leissing]

Westervelt equation

Westervelt equation: $(\partial_t^2 - \Delta)\Phi = \epsilon\partial_t[b\Delta\Phi + (a+1)(\partial_t\Phi)^2]$

Ways to obtain

- Original [Westervelt, 1963] - has two problems: ordinary acoustic pressure field is, in general, not governed by the Westervelt equation [Berntsen, 1989], and conditions of applicability are not so wide [Aanonsen, 1984]
- Modern [Aanonsen, 1984] - from Kuznetsov equation with the proper assumptions

Kuznetsov equation

$$\partial_t^2\widetilde{\phi} - \Delta\widetilde{\phi} = \epsilon\partial_t[b\Delta\widetilde{\phi} + (\nabla\widetilde{\phi})^2 + a(\partial_t\widetilde{\phi})^2]$$

Generalized potential

$$\Phi = \widetilde{\phi} + \epsilon\widetilde{\phi}\partial_t\widetilde{\phi}$$

Westervelt equation

$$(\partial_t^2 - \Delta)\Phi = \epsilon\partial_t[b\Delta\Phi + (a+1)(\partial_t\Phi)^2] + O(\epsilon^2)$$

Properties

Advantages

- the only model today, that is used for sound beam interaction
- it is possible to parametrize outcoming beam

Disadvantages

- energy conversion from high to low frequencies is poor
- lacks accuracy in sense of angle of collision
- bad convergence during numerical simulations (if one compares with other models)

Scheme of ultrasonic beam interaction

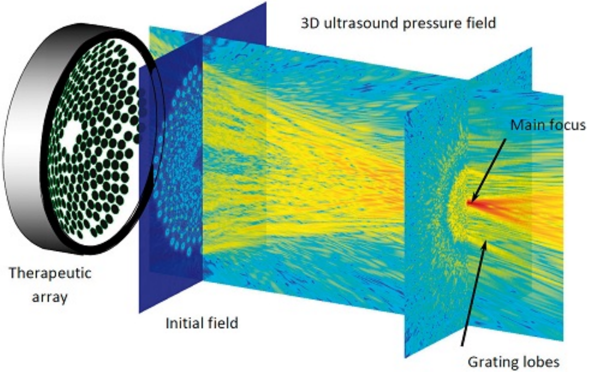


Figure: Scheme of ultrasonic beam interaction from the multi-element array [Courtesy of RCC of MSU]

Comparison of beam interactions in homogeneous and heterogeneous medium

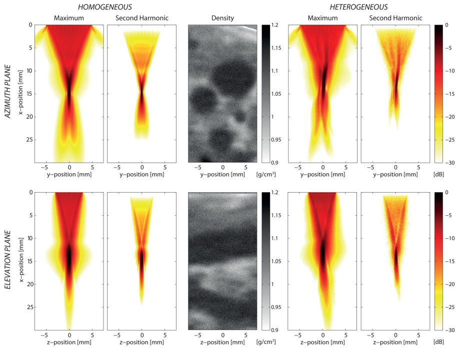
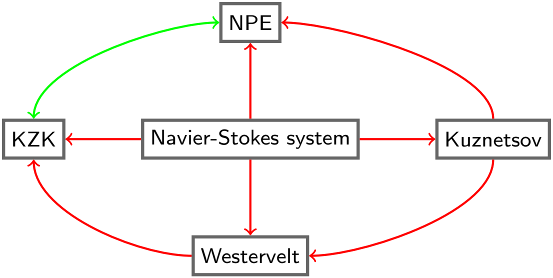


Figure: Normalized azimuth and elevation plane beam patterns generated by a clinical linear array ultrasound transducer [Courtesy of Treeby]

General scheme



Westervelt equation

- ∃! global solution for sufficiently small and regular data, it converges at an exponential rate as time tends to infinity [S. Meyer, M. Wilke, AMO, 2011];
- local and global well-posedness as well as exponential decay for the Westervelt equation with inhomogeneous Dirichlet/Neumann boundary [B. Kaltenbacher, I. Lasiecka, 2010],[B. Kaltenbacher, I. Lasiecka, S. Veljović, PNDETA, 2011]

(!) In this publications authors make no difference between generalized and usual potential. But boundary and initial conditions in general are not the same [Aanonsen, 1984].

NPE equation

This equation is widely enough used, but there is no proper analysis.

Kuznetsov equation

- ∃! global solution for the sufficiently small initial data (for arbitrary dimensions) for the corresponding Dirichlet boundary, it exponentially decays with time [S. Meyer, M. Wilke, EECT, 2013];
- proof, that ∃ optimal control, justification of the first order optimality conditions [C. Clason, B. Kaltenbacher, S. Veljović, JMAA, 2009].

Theorem

Let $T > 0$ be arbitrary. There exists $\rho_T > 0$ such that if:

$$E_{u,0}(0) + E_{u,1}(0) < \rho_T,$$

where

$$E_{u,0}(t) = \frac{1}{2}\|u_t(t)\|^2 + \|\nabla u(t)\|^2,$$

$$E_{u,1}(t) = \frac{1}{2}\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 + \|\Delta u\|^2, \text{ where } \|u\| \equiv \|u\|_{L_2(\Omega)},$$

$$E_{u,1}(0) \equiv \frac{1}{2}\{\left\|\frac{\Delta u_0 + \frac{\vartheta}{\rho_0}\Delta u_1 + \frac{\gamma-1}{\rho_0}u_1^2}{1 - \frac{\gamma-1}{\rho_0}u_0}\right\|^2 + \|\nabla u_1(t)\|^2 + \|\Delta u_0\|^2\},$$

where $(\rho(0), \rho_t(0)) = (u_0, u_1)$ – initial data.

then ∃! solution of Westervelt equation (u, u_t) (in a weak $H^{-1}(\Omega)$ sense) and such that:

$$u \in C([0,T]; H^2(\Omega)) \cap C^1([0,T]; H^1(\Omega)) \cap C^2([0,T]; L_2(\Omega)), \quad u_{tt} \in L_2((0,T); H^1(\Omega))$$

The said solution depends continuously (with respect to the topology generated by $E_{u,1}$) on the initial data.

Theorem

There exists $\rho > 0$, such that solutions corresponding to initial data $E_{u,1}(0) \leq \rho$ are global in time. By this we mean: There exists $\rho > 0$ and a positive constant $M > 0$, such that as long as $E_{u,1}(0) \leq \rho$, then $E_{u,1}(t) \leq M \ \forall t > 0$

Theorem

With ρ specified by Theorem above, there exists constants $\omega, \omega_1 > 0$, such that:

$$1. E_{u,0}(t) \leq C_\rho e^{-\omega t} E_{u,0}(0),$$
$$2. E_{u,1}(t) \leq C_\rho e^{-\omega_1 t} E_{u,1}(0).$$

Possible generalizations

- introduction of fractional derivative
- correction terms due to moving/nonhomogeneous medium

Acoustic attenuation

$$E = E_0 e^{-\alpha(\omega) z}$$

Empirical law

$$\alpha(\omega) = |\omega|^\gamma, \quad \gamma = 0 \quad \text{and} \quad \gamma = 2 \quad \text{– trivial cases}$$

- Covers only pure fluids, can be derived from some general assumptions
- Complex mediums like soft tissue or blood suffer from non-Gaussian attenuation

Solution

- usage of fractional Laplacian in the dissipative term
- a table with the empirical/theoretical data [Anese et al. [2013], Evans [1979], Omelyan et al. [2005]]

KZK equation in terms of fractional derivative [Courtesy of Prieur and Holm]:

$$2\partial^2_{\tau z}\rho_1 - \frac{\gamma+1}{2\rho_0}\partial^2_\tau \rho_1^2 - \partial^2_y \rho_1 - \frac{L_v - L_t}{\rho_0}\partial_\tau^{\Theta+1}\rho_1 = 0$$

where

$$L_v = \lambda^{\Theta-2}(\frac{4}{3}\eta + \xi), \quad L_t = \bar{\kappa}\lambda_{th}^{\Theta-2}(\frac{1}{C_V} - \frac{1}{C_P})$$

where $\lambda^{y-2}, \lambda_{th}^{y-2}$ were used to keep dimensionality.

And (given by Caputo):

$$\frac{d^\beta f}{dt^\beta} = \frac{1}{\Gamma(1-r)}\int_0^t \frac{1}{(t-\tau)^r}\frac{d^n}{dt^n} f(\tau)\ d\tau$$

where $0 \leq n-1 < \beta < n, \quad r = \beta - n + 1.$

Work in this direction

- All basic equations were reintroduced in terms of fractional derivatives [Prieur and Holm [2011]]
- It occurs, for example, that fractional Westervelt equation is good for a work with human tissues, when $1 \lesssim \gamma \lesssim 1.7$ [Ochmann and Makarov [1993], Szabo [1994]]
- Theoretical research: solutions of such equations and their validity [Holm and Nasholm [2011] Gan [2007]]), Bazhlekova [2014], Liu et al. [2013], Chen and Holm [2002]]

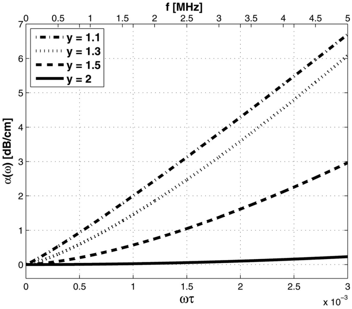


Figure: Attenuation $\alpha(\omega)$ as a function of $\omega\tau$ for different values of $\Theta \equiv \gamma_{att}$ [Courtesy of Prieur and Holm].