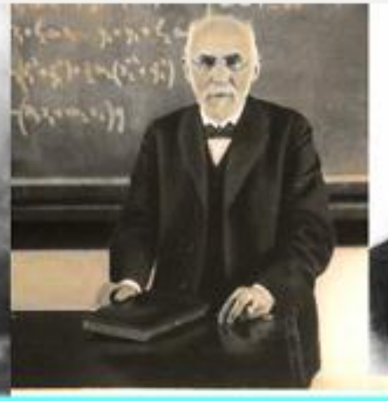
Groups in theoretical physics.
Inner and space-time symmetries.
Supergroups.

Content






1. *Lorentz and Poincare group.*
2. *Supersymmetry.*
3. *Symmetry breaking, topological defects.*



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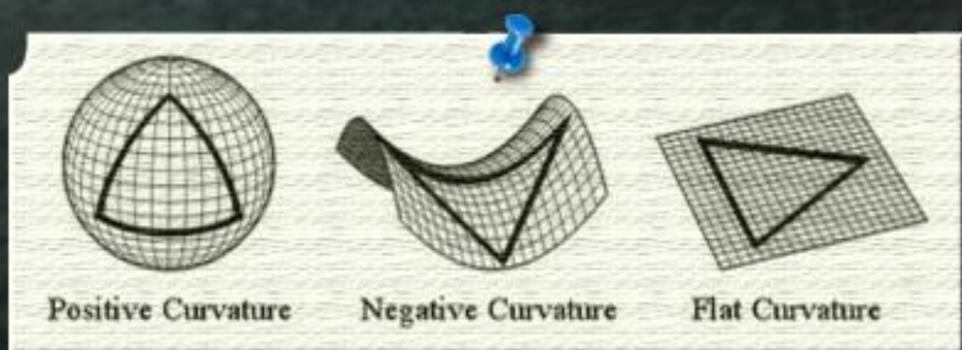
Lie derivative of metric:

$$\pounds_\xi g_{\mu\nu} = \xi^\eta \partial_\eta g_{\mu\nu} + g_{\eta\mu}\partial_\nu \xi^\eta + g_{\nu\eta}\partial_\mu \xi^\eta = 0 \qquad \textbf{(1)}$$

If space is homogeneous, we have $\frac{(n+1)}{2}$ Killings vectors ξ. For $3D$: 3 rotations + 3 translations.

***k*=1** ***k*=-1** ***k*=0**



Killing vectors produce infinitesimal transformations

$$x^{\mu} = x^{\mu} + \xi^{\mu}\delta t \qquad (2)$$

where t is parameter of one dimensional subgroup G_1 of the whole group G_n of transformations.

Let's define operators that correspond to these transformations

$$X_a = \xi^{\mu}_a \partial_{\mu} \qquad (3)$$

then can be built n-dimensional basis for the group G_n (algebra AG_n)

$$[X_a, X_b] = \left(\xi^{\mu}_a \partial_{\mu} \xi^{\nu}_b - \xi^{\mu}_b \partial_{\mu} \xi^{\nu}_a\right) \partial_{\nu} \qquad (4)$$

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If we do it for spaces with constant curvature, we will get the following algebra

$$
\begin{aligned}
[R_a, R_b] &= \varepsilon_{abc} R_c \\
[T_a, T_b] &= \frac{k}{\rho^2} \varepsilon_{abc} R_c \\
[R_a, T_b] &= \varepsilon_{abc} T_c
\end{aligned} \qquad (5)
$$

Let's consider a flat space and add time translation (Poincare group). In this case we have the following commutation relations

$$
\begin{aligned}
&[P^\mu, P^\nu] = 0 \\
&[M^{\mu\nu}, P^\gamma] = i\left(P^\mu \eta^{\nu\gamma} - P^\nu \eta^{\mu\gamma}\right) \\
&[M^{\mu\nu}, M^{\rho\sigma}] = i\left(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}\right)
\end{aligned} \qquad (6)
$$

where $P^\mu = (T_0, T_a)$; $R_a = \frac{1}{2}\varepsilon_{abc} M_{bc}$; $K_a = -M_{0a}$ - boost.

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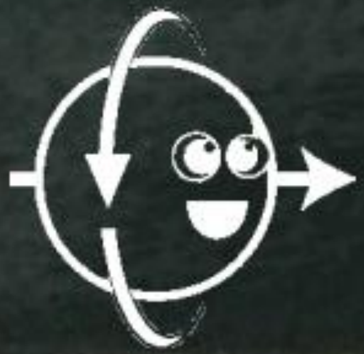
THERE CAN BE INTRODUCED OPERATORS $S_a = \frac{1}{2}(R_a + iK_a), J_a = \frac{1}{2}(R_a - iK_a)$, WHICH PROVIDE LOCAL ISOMORPHISM $SO(3,1) \backsim SU(2) \times SU(2)$

$$\begin{aligned} [S_a, S_b] &= i\varepsilon_{abc} S_c \\ [J_a, J_b] &= i\varepsilon_{abc} J_c \\ [J_a, S_b] &= 0. \end{aligned} \tag{7}$$

THERE ARE TWO CASIMIR OPERATORS: ONE FOR EACH $SU(2)$. ONE OF THEM IS RESPONSIBLE FOR MASS, ANOTHER FOR SPIN OF PARTICLE.



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Representation of Poincare group

Casimir operators are

$$C_1 = P^\mu P_\mu$$
$$C_2 = W^\mu W_\mu. \quad (8)$$

where $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ - is Paul-Ljubanski vector.

1) Massive case.
Let's choose $P^\mu = (m, 0, 0, 0)$, then we have
$W_0 = 0, W_a = -mR_a$.

That is why the state of massive particle can be described by the following *"ket"* vector in Hilbert space

$$|m, j, p^a, j_3 > \quad (9)$$

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2) Massless case.

Let's choose $P^{\mu} = (E, 0, 0, E)$, then we have $W_0 = -ER_3, W_1 = E(R_1 - K_2), W_2 = E(R_2 + K_1), W_3 = ER_3$. And the following commutations

$$[W_1, W_2] = 0$$
$$[W_3, W_1] = -iEW_2 \quad (10)$$
$$[W_3, W_2] = iEW_1$$

It forms a little subgroup $E(2,1)$.

In order to avoid infinite repr. $W_1 = W_2 = 0$. Then the *"ket"* vector for massless particle

$$|0, 0, p^a, \lambda > \quad (11)$$

Thank you for your attention!

to be continued...