Hyperbolic (Lobachevsky) space.

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Content

- 1. Definitions and mathematical tools.
- 2. Symmetries, (super)integrability and algebra of hyperbolic space.
- 3. Killing vectors and magnetic fields in hyperbolic space.
- 4. Applications: kinetic equations superconductivity and coherent states in hyperbolic space...

14/3/14

Definitions

Einstein summation:

$$\vec{x}\cdot\vec{y} = x^1y^1 + x^2y^2 + x^3y^3 = g_{i,j}x^iy^j, \text{ where}$$

$$g_{11} = g_{22} = g_{33} = 1$$

Definition of tensor:

$$Q^{j'}_{i'} = A^{j'}_{j} A^{i}_{i'} Q^{j}_{i}, \text{ where } A^{i'}_{i} = \frac{\partial x^{i'}}{\partial x^{i}}$$

Notation for derivative:

$$\frac{\partial f}{\partial \vec{x}} = \frac{\partial f}{\partial x^i} = \partial_i f = f_{,i}, \text{ but } Q_{j,i} \neq A^{i'}_{i} A^{j'}_{j} Q_{j',i'}$$

Covariant derivative:

$$D_i f = f_{;i} \text{ and } Q_{j;i} = A^{i'}_{i} A^{j'}_{j} Qj';i'$$

Lie derivative:

$$\pounds_\xi Q^j_i = \xi^k D_k Q^j_i + D_i \xi^k Q^j_k - D_k \xi^j Q^k_i$$

If space is homogeneous, we have $\frac{(n+1)}{2}$ Killings vectors. For 3D: 3 rotations + 3 translations.

$$\pounds_{\xi} g_{i,j} = \xi^k \partial_k g_{ij} + g_{kj} \partial_i \xi^k + g_{ik} \partial_j \xi^k = 0 \qquad (1)$$

All spaces with constant curvature are homogeneous: flat (k=0); elliptic (k=1); hyperbolic (k=-1).

14/3/14

Hyperbolic geometry (H_3):

- MORE SEPARABLE SYSTEMS [*Olevskii*, *Mat.Sb.*27, 379 – 426, 1950];
- SUPERINTEGRABILITY [*Cohl*, *Kalnins*, *Phys.A – Math.Theor*, 45 : 14, (2012)],

  [*Gibbons*, *Warnick*, *J.Geom.Phys.*57 : 2286 – 2315, (2007)].

KEPLER PROBLEM:

HAMILTONIAN $H = E = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \frac{\alpha}{\sqrt{q_1^2+q_2^2}}$,

ANGULAR MOMENTUM $\ell = q_1 p_2 - q_2 p_1$,

LAPLACE-RUNGE-LENZ VECTOR

$$\vec{e} = \{L_3, L_4\} = \left\{\ell p_2 - \frac{\alpha q_1}{\sqrt{q_1^2+q_2^2}}, -\ell p_1 - \frac{\alpha q_2}{\sqrt{q_1^2+q_2^2}}\right\}.$$

SUPERINTEGRABILITY IS REQUIRED $2n-1$ CONSTANTS.

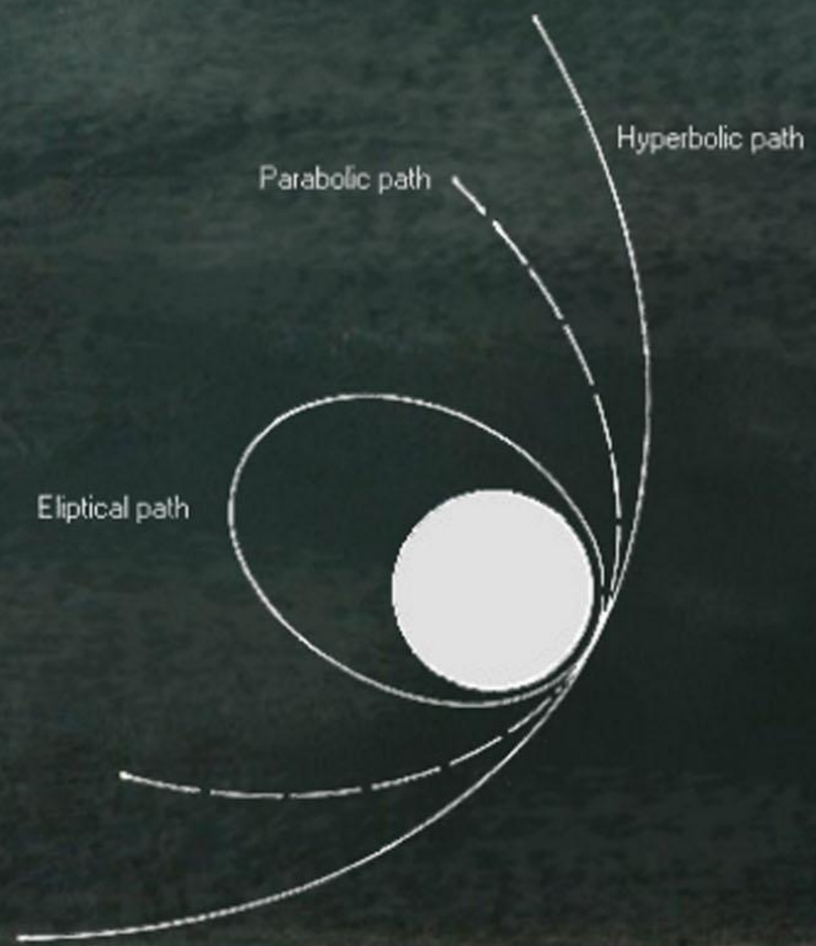
WE CAN CHOSE $\vec{e} = \{0, e_1\} \Rightarrow e_1^2 = 2\ell^2 E + \alpha^2$ AND

$$p_1 = -\frac{\alpha q_2}{\ell\sqrt{q_1^2+q_2^2}}, p_2 = \frac{e_1}{\ell} + \frac{\alpha q_2}{\ell\sqrt{q_1^2+q_2^2}}$$

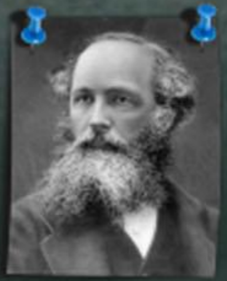
14/3/14

From definition of angular momentum

$$\frac{\ell^4}{\alpha^2} = \left(1 - \frac{e_1^2}{\alpha^2}\right) q_1^2 + \frac{2\ell^2 e_1}{\alpha^2} q_1 + q_2^2. \qquad (2)$$



14/3/14



Maxwell's equations in curved space.

ALL EQUATIONS OF MAXWELL CAN BE PUT IN TENSOR FORM

$$F^{ik}{}_{;k} = \frac{1}{\sqrt{|g|}}\partial_k\left(\sqrt{|g|}F^{ik}\right) = 0,$$

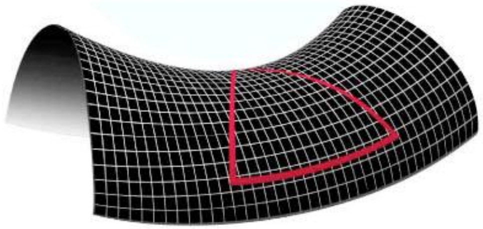
$$F_{ij;k} - F_{ki;j} - F_{jk;i} = 0, \quad (3)$$

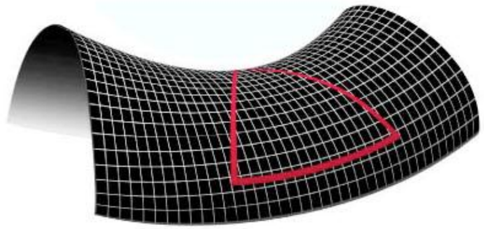
WHERE $F_{ik} = A_{i;k} - A_{k;i}$ - ELECTROMAGNETIC TENSOR.

FOR FLAT SPACE THERE IS A UNIFORM MAGNETIC FIELD

$$A_\phi = -\frac{Br^2}{2}.$$

Magnetic field in hyperbolic space





$$A_\phi = -\frac{Br^2}{2},$$

horospheric,

$$A_\phi = -B\left(\cosh\left(\frac{z}{\rho}\right) - 1\right),$$

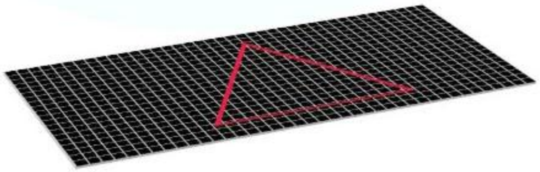
hyperbolic cylindrical,

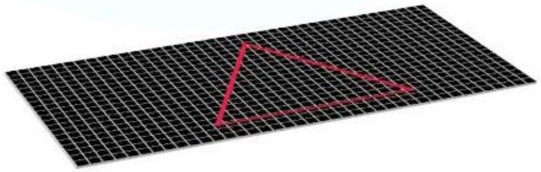
$$A_\phi = -B \ln \cosh\left(\frac{z}{\rho}\right).$$

cylindrical.

In the limit $\rho \longrightarrow \infty$

$$A_\phi = -\frac{Br^2}{2}$$





Symmetry properties

$$\pounds_\xi A_i = \xi^k D_k A_i + D_i \xi^k A_k$$

one vector (ξ_ϕ), one vector (ξ_ϕ), two vectors (ξ_ϕ, ξ_z)

$A_\phi = -\frac{Br^2}{2}$, *horospheric,*

$A_\phi = -B(\cosh\left(\frac{z}{\rho}\right) - 1)$, *hyperbolic cylindrical,*

$A_\phi = -B \ln\cosh\left(\frac{z}{\rho}\right)$. *cylindrical.*

In the limit $\rho \longrightarrow \infty$

$A_\phi = -\frac{Br^2}{2}$ two vectors (ξ_ϕ, ξ_z)

Magnetic field in hyperbolic space

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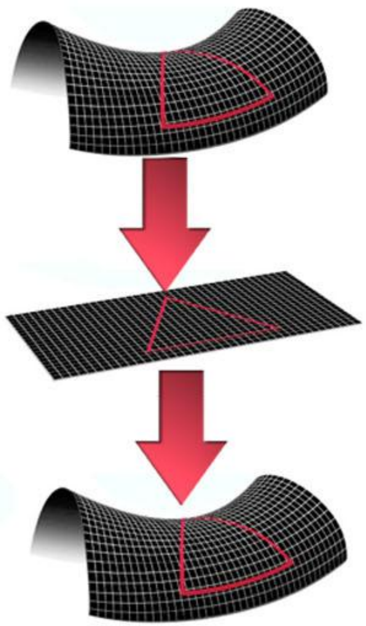

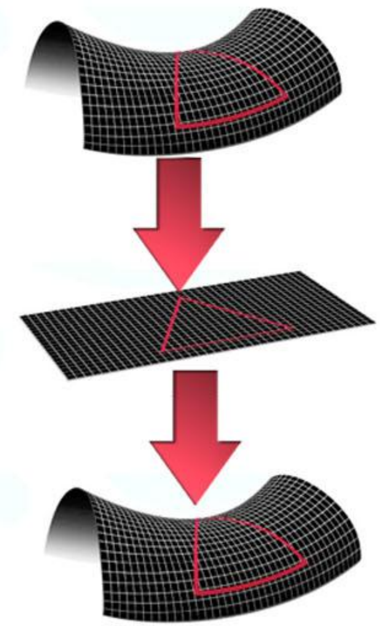

$$A_\phi = -\frac{Br^2}{2}$$

when $\rho < 0$

~~$A_\phi = -\frac{Br^2}{2}$~~, *horospheric,*

~~$A_\phi = -B(\cosh\left(\frac{z}{\rho}\right) - 1)$~~, *hyperbolic cylindrical,*

$A_\phi = -B \ln\cosh\left(\frac{z}{\rho}\right)$. *cylindrical.*

Kinetic equations

Solution of kinetic equations for thermodynamic equilibrium

$$f_a(x,p) = C_n exp\left(\frac{\mu_a - cu_\mu(p_a^\mu - \frac{e}{c}A^\mu)}{k_B T}\right). \quad (4)$$

where $u^\mu = \frac{\xi^\mu k_B T}{c}$ and $u_\mu u^\mu = 1$.
And ξ^μ is a Killing vector.

1. Some magnetic fields in hyperbolic space can poses smaller number of Killing vectors (symmetries), than their flat limit $\rho \longrightarrow \infty \Rightarrow$ kinetic equations can have less solutions than in the flat case.

[Yu.Kurochkin and I.Rybak, j-NPCS,16,1,65-71 (2013).]

14/3/14

Killing vectors produce infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \xi^{\mu}\delta t \quad (5)$$

where t is parameter of one dimensional subgroup G_1 of the whole group G_n of transformations.

Operators that correspond to these transformations

$$X_a = \xi_a^{\mu}\partial_{\mu} \quad (6)$$

then can be built n-dimensional basis for the group G_n (algebra AG_n)

$$[X_a, X_b] = \left(\xi_a^{\mu}\partial_{\mu}\xi_b^{\nu} - \xi_b^{\mu}\partial_{\mu}\xi_a^{\nu}\right)\partial_{\nu} \quad (7)$$

14/3/14

IF WE DO IT FOR SPACES WITH CONSTANT CURVATURE, WE WILL GET THE FOLLOWING ALGEBRA

$$[R_a, R_b] = \varepsilon_{abc} R_c$$
$$[T_a, T_b] = \frac{k}{\rho^2} \varepsilon_{abc} R_c \quad (8)$$
$$[R_a, T_b] = \varepsilon_{abc} T_c$$

WE CAN CHOOSE $T`_1 = T_1 + \frac{R_2}{\rho}$ AND $T`_2 = T_2 - \frac{R_1}{\rho}$ WE CAN GET FLAT SURFACE $[T`_1, T`_2] = 0$ WITH METRIC

$$ds^2 = \left(dx^2 + dy^2\right) e^{-2z/\rho} + dz^2. \quad (9)$$

Applications

2. It was shown that due to separation of variables in Laplace-Beltrami operator for the quantum particle in potential, which depends on horospherical variables, coherent states on the horosphere can be built up in Lobachevsky space by analogy with a flat space. Any problems, which are integrable in flat Euclidian plane, can be built up to integrable problem in three-dimensional Lobachevsky space.

[Yu. A. Kurochkin, I. Yu. Rybak, Dz. V. Shoukavy, DAN Belarus, 58, 5, 44-49 (2014)]

3. In hyperbolic space the Ginzburg-Landau model can have opposite to Meissner effect - attract magnetic flux.

[Kurochkin, Yu. A.; Rybak, I. Yu., Acta Phys Pol B, 45, 6, p. 1255 (2014)]

Thank you for your attention!