Topological solitons in scalar field theory

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Topological solitons in scalar field theory

- Introduction
- Sine-Gordon, ϕ^4 and ϕ^6 models
- Collisions (Scattering)
- Boundary collisions (ϕ^4 example)
- Demonstration
- Skyrme models (hopfions, baby Skyrmions, Skyrme crystals)
- Conclusions

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Solitons in nonlinear physics

Soliton: This is a solution of a nonlinear partial differential equation which represents a solitary travelling wave, which:

- Is localized in space
- Has a constant shape
- Does not obey the superposition principle.

Examples:

Optical fibres, rogue waves in ocean - NLSE

Josephson junctions - sine-Gordon model

Lattice QCD - caloron solutions

Superconductivity - Abrikosov-Nielsen-Olesen model

Topology

Topological charge

$$Q = \frac{1}{2}\int\limits_{-\infty}^{\infty} dx\,\frac{\partial\phi}{\partial x}$$

Non-toplogical solitons

- KdV solutions
- Lump-solution in various polynomial models
- Oscillons

Topological solitons

- Kinks, domain walls, vortices in various models
- Skyrmions
- Hopfions

Sine-Gordon, ϕ^4 and ϕ^6 models

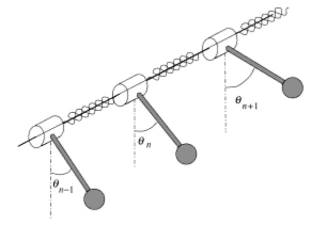
sine-Gordon model

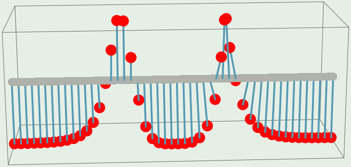
Field equations

$$\frac{\partial^2 v}{\partial t\,\partial t} - \frac{\partial^2 v}{\partial x\,\partial x} + \sin(v) = 0$$

$$v(x,t) = 4\arctan e^{(-\sqrt{-1+k^2}t+kx)}$$

A simple mechanical example: the chain of coupled pendula



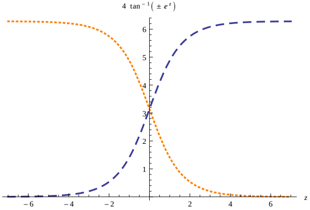


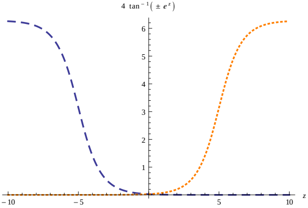
sine-Gordon model

Hamiltonian and equations of motion

$$H = \sum_{n}^{N} \frac{I}{2}\left(\frac{d\theta_n}{dt}\right)^2 + \frac{C}{2}\left(\theta_n - \theta_{n-1}\right)^2 + mgl(1 - \cos\theta_n)$$

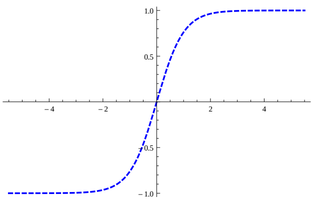
$$I\frac{d^2\theta_n}{dt^2} - C(\theta_{n+1} + \theta_{n-1} - 2\theta_n) + mgl\sin\theta_n = 0$$

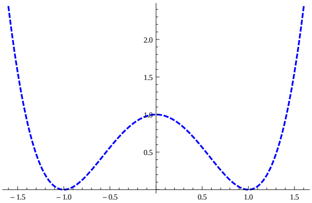




How to make an experimental setup for sine-Gordon kinks with a belt and pins?

ϕ^4 potential model





Lagrangian and equation of motion

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(\phi^2 - 1\right)^2$$

$$\partial_{tt}\phi - \partial_{xx}\phi + 2\phi(\phi^2 - 1) = 0$$

Static solution (kink/antikink)

$$\phi = \tanh(A(\pm x + x_0))$$

Modes of ϕ^4 kink

Linear oscillations on the background

$$\phi = \phi_K + \delta\phi$$

where $\delta\phi$ can be expanded in a set:

$$\delta\phi = \sum_{n=0}^{\infty} C_n(t)\eta_n(x)$$

Eigenfunctions

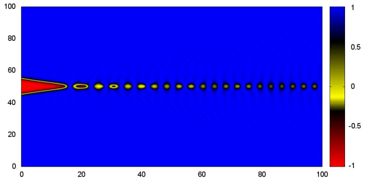
$$\eta_0(x) = \frac{1}{\cosh^2\ x}$$
$$\eta_1(x) = \frac{\sinh x}{\cosh^2\ x}$$
$$\eta_k(x) = e^{ikx}(3\tanh^2\ x - 3ik\tanh x)\quad - e^{ikx}(1+k^2) + \mathrm{C.C.}.$$

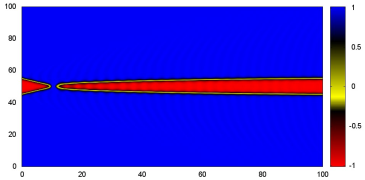
ϕ^4 kink-antikink collisions

Properties

- Annihilate or repel
- Produce oscillon state after annihilation
- Have bounce windows structure
- Fractal structure

Bounce windows and fractal structure refers to the energy exchange mechanism between translational and internal modes





Bounce windows structure

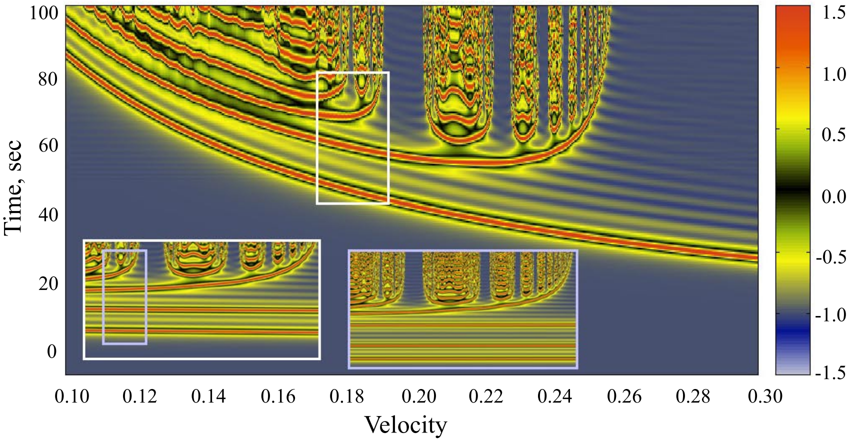
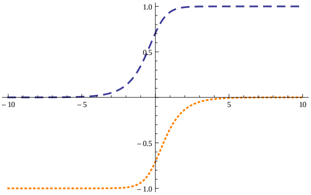
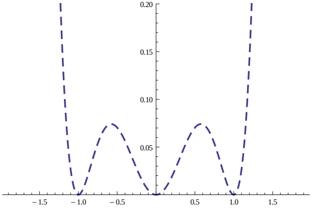


Figure: Bounce window structure of $K\bar{K}$ collision in a usual ϕ^4 case

ϕ^6 potential model





Lagrangian and equation of motion

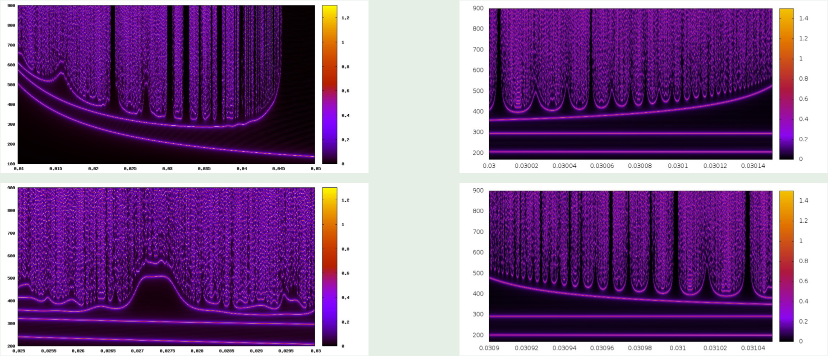
$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\phi^2\left(\phi^2 - 1\right)^2$$

$$\partial_{tt}\phi - \partial_{xx}\phi + 2\phi^3(\phi^2 - 1) + \phi(\phi^2 - 1) = 0$$

Static solution (kink/antikink)

$$\phi = \pm\sqrt{\frac{1 \pm \tanh x}{2}}$$

Quick glance on ϕ^6 window structure



- No internal mode. Energy exchange is due to continuous spectrum
- Missing window is probably an interference effect (no theory yet)
- Collective coordinate model is under construction (no theory yet)

Coupled ϕ^4 model

Lagrangian of the coupled two-component model can be written as:

$$L = \frac{1}{2}[(\partial_t\phi_1)^2 - (\partial_x\phi_1)^2 - (\phi_1^2 - 1)^2] + \frac{1}{2}[(\partial_t\phi_2)^2 - (\partial_x\phi_2)^2 - (\phi_2^2 - 1)^2] + \kappa\phi_1^2\phi_2^2$$

Field equations:

$$\begin{cases} \partial_t^2\phi_1 - \partial_x^2\phi_1 + 2\phi_1(\phi_1^2 - 1) - 2\kappa\phi_1\phi_2^2 = 0 \\ \partial_t^2\phi_2 - \partial_x^2\phi_2 + 2\phi_2(\phi_2^2 - 1) - 2\kappa\phi_2\phi_1^2 = 0 \end{cases}$$

Wess-Zumino model with two coupled Majorana spinor fields
Montonen-Sarker-Trullinger-Bishop model with two scalar coupled fields

Static configuration (numerical)

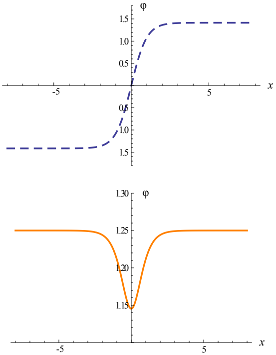


Figure: Static configuration for $\kappa = 0.5$

A. Halavanau, T. Romanczukiewicz, Ya. Shnir - Resonance structures in coupled two-component ϕ^4 model, Physical Review D 86, 085027

Different collision channels

Topological charge

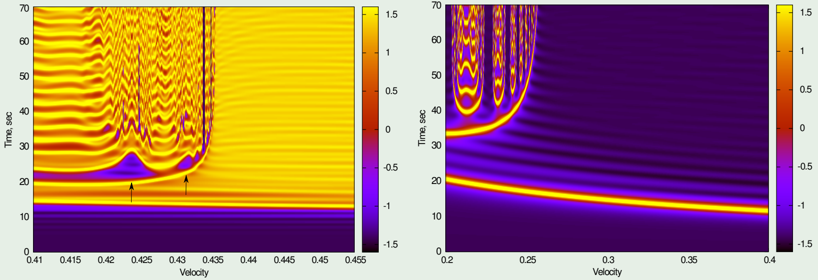
$$Q_i = \frac{1}{2}\sqrt{1-\kappa} \int_{-\infty}^{\infty} dx \, \frac{\partial \phi_i}{\partial x}, \qquad i = 1, 2$$

Interesting channels

$$(-1,0) + (1,0) \rightarrow \begin{cases} (0,0) \\ (0,-1) + (0,1) \\ (-1,0) + (1,0) \end{cases}$$

$$(-1,1) + (-1,1) \rightarrow \begin{cases} (0,0) \\ (-1,1) + (-1,1) \end{cases}$$

Topological flipping and double kink configuration



Kink-lump collision (left) and double kink collision (right). Numerical errors prove FDM is not sufficient for the problem and spectral methods are to be used.

Applying boundary conditions on the equations of motion

following the principle of stationary action, one can obtain the equations of motion for the bulk and the boundary (assuming $M(\phi,\phi_t) = B(\phi) - A(\phi,\phi_t)$):

$$\left(\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \phi_t} - \frac{d}{dt}\frac{\partial L}{\partial \phi_x}\right) = 0$$

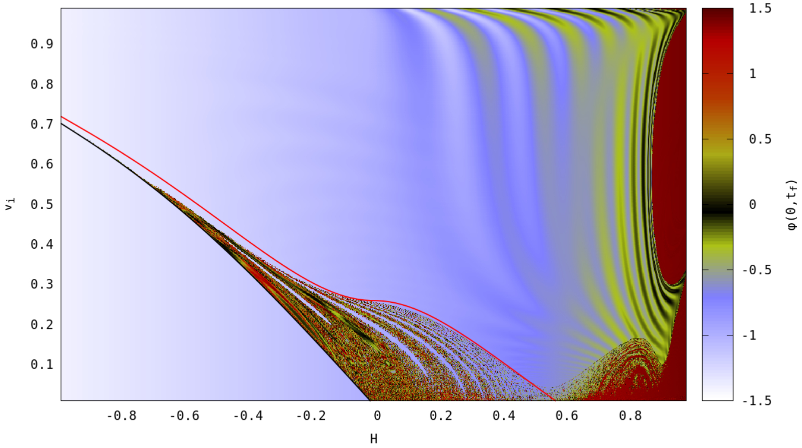
and for the boundary point $x = x_b$

$$\left(\frac{\partial L}{\partial \phi_x} - \frac{\partial M}{\partial \phi} + \frac{d}{dt}\frac{\partial L}{\partial \phi_t}\right) = 0$$

$\phi_x(0,t) = H_b = const$ from where $x_0 = x_b + \cosh^{-1}\left(\sqrt{\frac{1}{H}}\right)$ and few modes survive from collective spectrum:

$$\eta(x) = e^{ik(x-x_0)}\left(-1 - k^2 - 3ik\tanh(x-x_0) + 3\tanh(x-x_0)^2\right)$$

Neumann boundary (applied magnetic field)



by courtesy of T. Romanczukiewicz

Beyond 1+1...

Derrick's theorem shows that stationary localized solutions to a nonlinear wave equation or nonlinear Klein–Gordon equation in dimensions three and higher are unstable.

Skyrme model

$$E = \frac{1}{32\pi^2\sqrt{2}} \int \left(\partial_i\phi\partial_i\phi + \frac{1}{2}(\partial_i\phi\partial_i\phi \times \partial_i\phi\partial_i\phi)(\partial_i\phi\partial_i\phi \times \partial_i\phi\partial_i\phi) \right)$$

The first term in the energy is that of the usual O(3) sigma model and the second is a Skyrme term, required to provide a balance under scaling and hence allow solitons with a finite non-zero size.

- Skyrmions, baby Skyrmions
- Hopfions
- Skyrme crystals

Isorotating baby Skyrmions

Lagrangian and topological charge

$$L = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{4}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 - U(\vec{\phi})$$

$$B = \frac{1}{4\pi}\int \vec{\phi}\cdot\partial_1\vec{\phi}\times\partial_2\vec{\phi}\; d^2x$$

Different potential choices

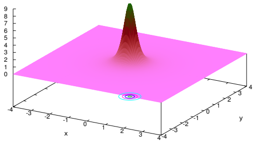
- $U(\vec{\phi}) = \mu^2[1-\phi_3]$
- $U(\vec{\phi}) = \mu^2[1-\phi_3^2]$
- $U(\vec{\phi}) = \mu^2[1-\phi_3^2][1-\phi_1^2]$

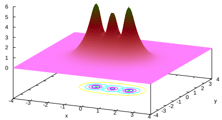
Rotation invariance

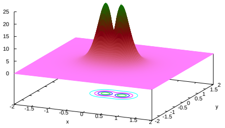
$$(\phi_1 + i\phi_2) \to (\phi_1 + i\phi_2)e^{i\omega t}$$

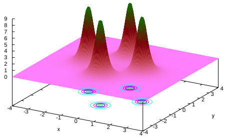
Isorotating baby Skyrmions

A. Halavanau, Yakov Shnir - Isorotating Baby Skyrmions, Phys. Rev. D 88, 085028 (2013)









Faddeev-Skyrme model

Lagrangian and mapping

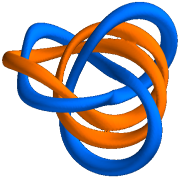
$$L = \frac{1}{32\pi^2}\left(\partial_\mu\phi^a\partial^\mu\phi^a - \frac{\kappa}{4}(\varepsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c)^2 + \mu^2[1-(\phi^3)^2]\right)$$

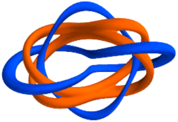
$R^3 \to C^2$ mapping

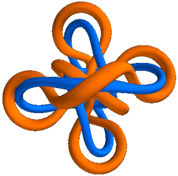
$$W = \frac{p(Z_1, Z_0)}{q(Z_1, Z_0)} = \frac{Z_1^\alpha Z_0^\beta}{Z_1^a + Z_0^b}$$

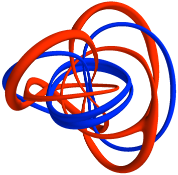
Topological charge (Hopf charge)

$$Q = \alpha b + \beta a$$









Massive and isospinning hopfions (work in progress). arXiv:1301.2923 [hep-th], J. Jaykka, Ya. Shnir, D. Harland and M. Speight
A. Acus, A. Halavanau, E. Norvaisas and Ya. Shnir - Hopfion canonical quantization, Physics Letters B 711 (2012) [hep-th]

Collaborations and resources

Existing collaborations

- University of Oldenburg, with Ya. Shnir
- Uniwersytet Jagiellonski, with T. Romanczukiewicz
- Durham University, with P. E. Dorey and P. Sutcliffe
- Vilnus University, with A. Acus
- Stockholm University, with J. Jaykka

Resources

- Mathematica 8
- SKIF supercomputer, Belarusian State University
- Condor computing cluster, Durham University
- Over 2000 hours of computing time

Questions? Comments?